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The Four-Color Theorem and Chromatic Numbers of Graphs

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Abstract

We study graph colorings of the form made popular by the four-color theorem. Proved by Appel and Haken in 1976, the Four-Color Theorem states that all planar graphs can be vertex-colored with at most four colors. We consider an alternate way to prove the Four-Color Theorem, introduced by Hadwiger in 1943 and commonly know as Hadwiger's Conjecture. In addition, we examine the chromatic number of graphs which are not planar. More specifically, we explore adding edges to a planar graph to create a non-planar graph which has the same chromatic number as the planar graph which we started from.

Introduction

The four-color problem deals with coloring maps, and the problem itself is reasonably simple to describe. In order to clearly distinguish countries on a map, we need to color neighboring countries with different colors. Then, in order to color the whole map in this fashion, what is the fewest number of colors we can use? More generally, what is the fewest number of colors we need to insure that we can color any map in this fashion using only that number of colors? To visualize this, consider coloring a map of the United States (Fig. 1) with the fewest number of colors in such a way that states which share a border are not the same color. The four-color problem asks: "Can every map be colored with, at most, four colors in such a way that neighboring countries are colored differently?"

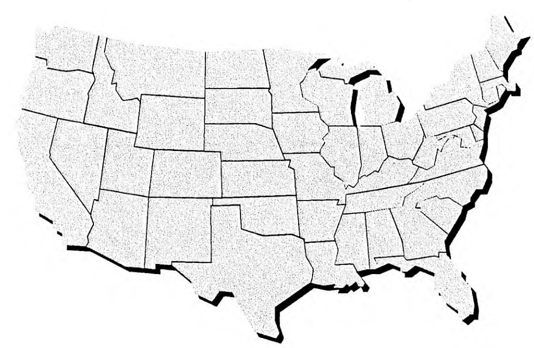

Fig. 1: A sample map to which the four color theorem can be applied.

First asked in 1852, this question is seemingly simple, and thus it would be likely that a proof of such a concept would also be simple. However, the problem stumped many mathematicians and went unsolved for over 120 years. A proof was eventually obtained, but over 1,000 computer hours were required to process the multitude of possible cases. Since the proof of the four-color problem in 1975, it is now referred to as the four-color theorem.

Four-Color History

Approximately one hundred fifty years ago Francis Guthrie described to his brother Frederick a proof that four colors are sufficient to color any map so that countries sharing a border are assigned different colors. Frederick then passed this idea on to his teacher, Professor

Augustus De Morgan, but the structure and content of the 'proof' that Francis gave to his brother is unknown [1, 2]. The documented origin of the four-color problem is traced to a letter from De Morgan to his friend Sir William Hamilton on October 23, 1852. In this letter he describes the classroom interaction where Frederick Guthrie posed his brother's four-color proposition and a few points of justification. In this letter De Morgan comments, "The more I think of it, the more evident it seems" [2]. As much as he tried to solve this four-color problem and inspire others to think about the solution, De Morgan made little progress on a proof by the time of his death in 1872.

The four-color problem first became popular in 1878 when Arthur Cayley mentioned it before the Royal Society [3]. Shortly after, supposed "proofs" of the four-color problem began to circulate. In one attempt to solve this problem, Kempe decided to draw the map in a different way. He observed that by placing a piece of tracing paper over a map, drawing one point on each country, and then connecting points whenever the corresponding countries shared a border, a "linkage" diagram is formed (Fig. 2). Such a diagram is now called a graph, and some sources site this as the beginning of graph theory. I find the four-color problem very interesting because some great contributions in various areas of math seem to stem from ideas which emerged during failed attempts at a proof. As Robin Wilson describes, "the four color problem itself may not be part of the mathematical mainstream, but the advances it has inspired are playing an increasingly important role in the evolution of mathematics" [2].

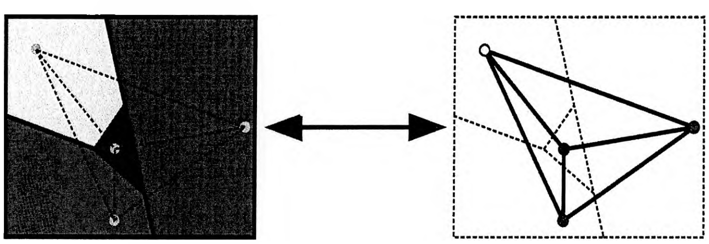

Fig. 2: An illustration of how to convert a map of regions into a "linkage" diagram, or graph.

Graph Theory Basics

Before continuing, we need to have a good understanding of the basic definitions and concepts of graph theory. A graph, sometimes referred to as a simple graph, $G = (V, E)$ is composed of a set *V* of points, called vertices, and a set *E* of distinct pairs of vertices, called edges. For example, consider a graph $G = (V, E)$ with vertex set $V = \{a, b, c, d\}$ and edge set $E = \{\alpha, \beta \mid \alpha = \{a, b\}, \beta = \{c, d\}\}$. Then $G = (V, E)$ will look like the diagram to the right (Fig. 3). In this example we have a graph $G = (V, E)$ such that $a, b \in V$ and $\alpha = \{a, b\} \in E$, so we say that α joins *a* and *b*. Also, *a* and α are incident and similarly, *b* and α are incident.

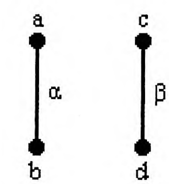

Fig. 3: A sample graph, $G = (V, E)$

In addition, we label *a* and *b* the vertices of the edge α. To draw a geometric diagram of *G*, a vertex-point is drawn to correspond with each vertex in *V* and a simple curve is drawn corresponding with each edge in *E*, called an edge-curve. An edge-curve α passes through a vertex-point *x* only if *x* is a vertex of the edge α.

When observing aspects of a graph, the number of vertices, *n*, in the vertex set of *G* is called the order of the graph *G*, denoted $|G| = n$. The degree of a vertex *x* in *G*, written $\deg(x)$, is the number of edges which are incident with *x*. If we have a graph of order *n* such that $\deg(x) = n - 1$ for all *x* in *G*, then *G* is a complete graph on n vertices, notated K_n.

Now we will describe different ways in which these vertices can be connected within a graph. A path is a sequence of *m* edges of the form $\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{m-1}, x_m\}$. We say this path joins the vertices x_0 and x_m. A path can also be denoted $\{x_0 - x_1 - x_2 - \ldots - x_m\}$. If $x_0 = x_m$, the path is closed. If $x_0 \neq x_m$, the path is open. When the vertices of the path are distinct, except for x_0 potentially being equal to x_m, then the path is referred to as a chain. A closed chain, where $x_0 = x_m$, is called a cycle.

A graph is connected if for every pair of vertices x and y there is a path joining x and y. More specifically, the existence of such a path implies that there exists a chain joining x and y. Otherwise, G is disconnected, and there exists at least one pair of vertices x and y for which there is no way to get from x to y by following the edges of the graph. A sub-graph $G' = (U, F)$ of a graph $G = (V, E)$ is a graph such that U is a subset of V and F is a subset of E such that the vertices of each edge in F are elements of U. An induced sub-graph of G includes a subset of vertices, U, and all of the edges that join these vertices, denoted G_U.

In order to discuss different ways of coloring a graph, first we need to understand what graph coloring entails. This topic deals primarily with vertex colorings. A vertex coloring of G is an assignment of a color to each of the vertices of G in such a way that adjacent vertices are assigned different colors. When we chose these colors from a set of k colors we call it a k-vertex-coloring, or k-coloring, even if all k colors are not used. If G has a k-coloring, then we say G is k-colorable. With these definitions it is clear that a k-colorable graph will also have an m-coloring for any $m \geq k$. However, the smallest k for which G is k-colorable is called the chromatic number of G, denoted by $\chi(G)$.

The idea of planarity is also a key aspect of graphs and essential to understanding the Four Color Theorem. By definition, a graph G is planar if it can be drawn such that two edges intersect only at vertex points. If this is not the case, the graph is non-planar. This characteristic may not be easily seen in all graphs of all sizes, so various theorems help in distinguishing the planarity of a graph. First, we have Euler's Formula:

Theorem P.1: Let G be a plane-graph of order n with e edges and assume that G is connected. Then the number r of regions into which G divides the plane satisfies the equation $r = e - n + 2$.

In addition to Euler's Formula, there are two more helpful proofs about planar graphs:

Theorem P.2: Let *G* be a connected planar graph. Then there is a vertex of *G* whose degree is at most *5*.

Theorem P.3: A graph *G* is planar if and only if it does not have a sub-graph which contracts to a K_5 or a $K_{3,3}$.

Though the individual concepts that form the basis of graph theory may not seem extremely complicated, the conjectures that arose from them were considered the most famous unsolved problem for over one hundred years.

The First Four-Color "Proof"

Now, to resume discussion of the coloring problem, it is important to look at the work by Kempe. In 1878 Kempe published a proof by mathematical induction to solve the Four-Color Conjecture. His induction hypothesis assumes that there exists a graph of order n – 1 that is four-colorable. This graph $G - \{v_0\}$ was obtained by removing a vertex v_0 of degree ≤ 5 from a planar graph G, and we know such a point exists in G because of Theorem P.2. Now, the inductive step involves adding this vertex v_0 back into the graph G and finding a method to recolor the vertices of G such that knowing the graph $G - \{v_0\}$ is four-colorable implies G must be four-colorable.

To do this, assume $G - \{v_0\}$ is four-colored arbitrarily[(1)] with red, blue, yellow, and green.

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[1] *To avoid repetition I will note here that any initial assignment of a color to a point v_i in $G - \{v_0\}$, a where $i > 0$ is an integer, is arbitrary. The numbers 1, 2, 3, 4 would form the same proof, and the reader could be instructed to "paint-by-number" in the end with any colors choices so long as every "1" was the same color, every "2" was the same color, and the same for 3, and 4 as well. I feel this introduced too many unnecessary numbers and variables into the proof that will ultimately only further confuse a reader not familiar with the topic. If we use an arbitrary assignment of red to an initial point, it would be perfectly valid to use any shade of orange, purple, or any other hue to color the red vertices, so long as the color is not already used in the graph and every red vertex is assigned this new color.*

Kempe did this step in five cases:

Case 1: Let *deg*(v_0) = 1. Then v_0 is incident to exactly one other vertex, v_1. Let v_1 be a red vertex in graph G – {v_0}. Then v_0 can be assigned any of the remaining three colors, and the rest of the vertices in G can be colored exactly the same as they were in G – {v_0}, and we have a four-coloring of G.

Case 2: Let *deg*(v_0) = 2. Then v_0 is incident to exactly two other vertices, v_1 and v_2. Either both v_1 and v_2 are colored red, or v_1 is red and v_2 is blue. Then, in either case, v_0 can be colored either green or yellow, and the rest of the vertices in G can be colored exactly the same as they were in G – {v_0} to obtain a four-coloring of G.

Case 3: Let *deg*(v_0) = 3. Then v_0 is incident to exactly three other vertices, v_1, v_2 and v_3. Now there are three coloring option: all three vertices are red, two vertices are red and one vertex is blue, or v_1 is red, v_2 is blue, and v_3 is green. In all three cases, v_0 can be colored yellow. Then assigning the rest of the vertices the same colors as in G – {v_0}, we get a four-coloring of G.

The remaining two cases involve an in-depth analysis of coloring. Regretfully, this will not explain Kempe's argument in full detail, but it will provide a more concise and understandable explanation of Kempe's reasoning for each the two cases. The diagrams below (Fig. 4) show graphical examples for *deg*(v_0) = 4 and for *deg*(v_0) = 5.

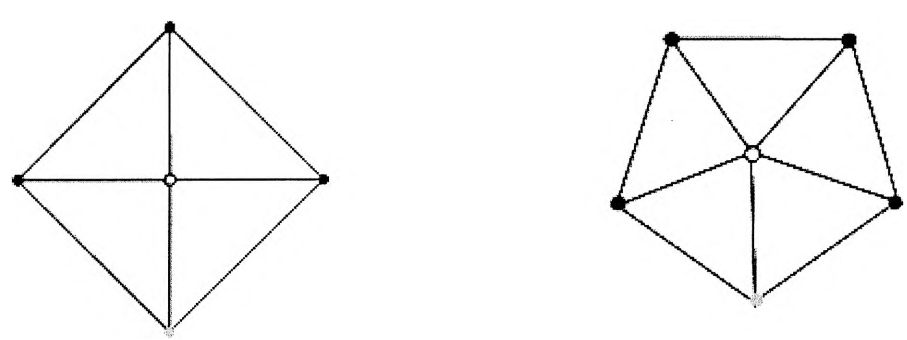

Fig. 4: An example graph for deg(v_0)=4 is shown of the left, and a graph for deg(v_0)=5 is on the right.

To begin Case 4, first let $deg(v_0) = 4$. Then v_0 is incident to exactly four other vertices, v_1, v_2, v_3 and v_4. There are many different ways in which we can assign color the vertices, however we describe these coloring in two sets. The first set is all coloring of v_1, v_2, v_3 and v_4 such that at most three colors are used. In all of these cases, there is one free color with which we can color v_0. The second set contains the coloring such that v_1 is red, v_2 is blue, v_3 is green, and v_4 is yellow. Then we must find a way to re-color the vertices adjacent to v_0 using only three colors. Similarly with $deg(v_0) = 5$, we have v_1 is red, v_2 is blue, v_3 is green, v_4 is yellow, and v_5 is blue. Thus, this graph also needs a method to recolor the vertices adjacent to v_0.

The method Kempe used is now known as Kempe's chain argument. The flaws in this method will be addressed later, but for now the argument will follow as Kempe described in his original proof. To begin this method, we find two non-adjacent vertices, such as the red and green vertices. These two vertices are starting points of red-green Kempe chains and define a section of the graph that is colored in only red and green. Kempe's argument goes on to show that one of two things must occur. The first option is that the branch starting at the green vertex and the branch starting at the red vertex never meet up. Then the green vertex can be colored red. This frees the color green for the center vertex. Otherwise, the branches starting at the red and green vertices do link up. Then the argument claims that the blue-yellow branch starting with the blue vertex which is adjacent to both the red and green vertices will be "cut-off" at some point by the red-green chain, and thus this vertex can be colored yellow. For $deg(v_0) = 4$ we are done. For $deg(v_0) = 5$ the proof continues to state that there exists a red-yellow chain that acts similarly on the other blue vertex, allowing that vertex to be colored green and allowing v_0 to be colored blue.

This proof of the Four-Color Theorem was accepted by the worldwide mathematic community for 11 years. In 1889, however, Heawood presented a counter-example map [4].

The map did not contradict the Four-Color Theorem; it only contradicted the chain argument Kempe used in the case where $deg(v_0) = 5$. Though it is perfectly valid to interchange the color of one vertex dependent on the nature of a Kempe chain in a graph, the flaw in Kempe's proof was in assuming that he could interchange and re-color two non-adjacent vertices simultaneously. Since Kempe's chain argument and proof was valid for $deg(v_0) \leq 4$, the proof was slightly modified and published as the 5-Color Theorem. It was still a good advancement in the coloring problem, and the methods and ideas Kempe used in his multiple attempts at the Four-Color Theorem greatly contributed to many areas of math.

In fact, Kempe's ideas were the inspiration behind the idea of reducible sets, which were eventually used to prove the Four-Color Theorem. Unfortunately, it took approximately a century after the conjecture was posed until the necessary tools were developed to create a conclusive proof of the theorem. In 1976 Wolfgang Haken and Kenneth Appel found the solution, primarily because they utilized 1,000 hours of computer time to come up with the proof, and the data provided at the time was essentially unverifiable at the time. Even now, running a computer program to analyze the number of configurations they used would not be a fast process. This method of proof brought about very mixed reactions from the mathematical community. Although the accomplishment of finding this solution was great, many mathematicians remain skeptical of the 'proof' since they cannot directly check the argument by hand. Regardless, they did find a solution, and their method of choice has kept the Four-Color Problem alive in the heart of mathematicians still searching for a simply executed proof to such a simply stated problem.

The history behind the Four-Color Theorem brings up two questions. First, is there a simple or more logical way to solve the Four-Color Problem? Secondly, can we classify the chromatic number of non-planar graphs? I find both questions interesting and decided to use both questions as sources of inspiration for my research.

Alternative Proof Methods

While researching graph theory and the history of the Four-Color Theorem, *Hadwiger's Conjecture* left an impression on me. Posed by Hadwiger in 1943, the conjecture states, "A graph G whose chromatic number satisfies $\chi(G) \geq p$ can be contracted to a K_p. Equivalently, if G cannot be contracted to a K_p then $\chi(G) < p$" [5]. This conjecture is know to be true for $p \leq 4$, though the proof of $p = 4$ is quite difficult to solve. This conjecture is also extremely relevant because it is known that Hadwiger's conjecture holds for $p = 5$ if and only if every planar graph has a four-coloring. That is, proving Hadwiger's conjecture for $p = 5$ is equivalent to solving the Four-Color Theorem [6].

Hadwiger's conjecture seems like a good foundation for developing a more logic-based proof of the Four-Color Theorem that does not rely so heavily on the use of computers. Right now I am working on the proof of Hadwiger's conjecture for $p = 4$. Although it is known to be true, the proof is not very accessible, thus, in order to see how it is done, I must do it myself. Hopefully, the proof of the conjecture for $p = 4$ will give me a good understanding of the techniques needed for the proof of Hadwiger's conjecture for $p = 5$. I have included my proof of Hadwiger's Conjecture for $p = 4$, as well as a few definitions and theorems from Brualdi's text book on combinatorics [5] which are necessary to complete the proof.

Necessary Definitions/Theorems for Hadwiger Proof

Definition - A graph is called color-critical provided each sub-graph obtained by removing a vertex has a smaller chromatic number.

Theorem A.1: If $G=(V,E)$ is a color-critical graph, then we can say the following:

(a) $\chi(G_{V-\{x\}})=\chi(G)-1$ for every vertex $x\in G$.

(b) G is connected.

(c) Each vertex of G has degree at least equal to $\chi(G)-1$.

(d) G does not contain an articulation set U such that G_U is a complete graph.

(e) Every graph H has an induced sub-graph G such that $\chi(G)=\chi(H)$, and G is color-critical.

Theorem A.2: Let $p\geq 3$ be an integer. If G is a graph such that each vertex in G has degree at least $p-1$, then G contains a cycle of length greater than or equal to p.

Corollary A.3: Let $p\geq 3$ be an integer. If $\chi(G)=p$, then G contains a cycle of length greater than or equal to p.

Proof of Hadwiger's Conjecture for p=4

Let H be a graph such that $\chi(H)=4$. Then H has an induced sub-graph $G=(V,E)$ such that $\chi(G)=4$ and G is color-critical (by Theorem A.1). Then $\forall v\in V$, $\deg(v)\geq 3$, and G does not contain an articulation set (by Theorem A.1). This implies that G contains a cycle of length ≥ 4 (by Theorem A.2).

Let $Y=x_0-x_1-\cdots-x_k$ be the cycle of largest length in G. Then, $|Y|\geq 4$. We know every $x_n\in Y$ is adjacent to two other elements of Y, x_{n+1} and x_{n-1}. Then x_n must be adjacent to at least one other vertex of G because $\deg(x_n)\geq 3$; call this point z. Now we have $x_{n-1}-x_n-x_{n+1}$ and x_n-z. Suppose z is not an element of a chain connecting x_n to another $x_j\in Y$. Then removing x_n will disconnect z from Y. So, x_n is an articulation vertex. $\Rightarrow\Leftarrow$.

This is a contradiction because G does not contain any articulation vertices. Therefore, G can be contracted to $G' = (V', E')$, where $V' = \{x_0, x_1, \ldots, x_k\}$. That is, G' only contains the vertices of the largest cycle, each x_n is adjacent to at least 3 other vertices of G'.

Case 1: Suppose there exist 4 vertices, $x_a, x_b, x_c, x_d \in G'$ such that $0 \leq a < b < c < d$, $x_a - x_c$, and $x_b - x_d$. Looking at the diagram below (Fig. 5), we can see that this forms a cross. Consider any vertex $x_p \in G'$. If $a < p \leq b$, contract x_p to x_b. If $b < p \leq c$, contract x_p to x_c. If $c < p \leq d$, contract x_p to x_d. If $p \leq a$ or $p > d$, contract x_p to x_a. We have now contracted G' to a K_4.

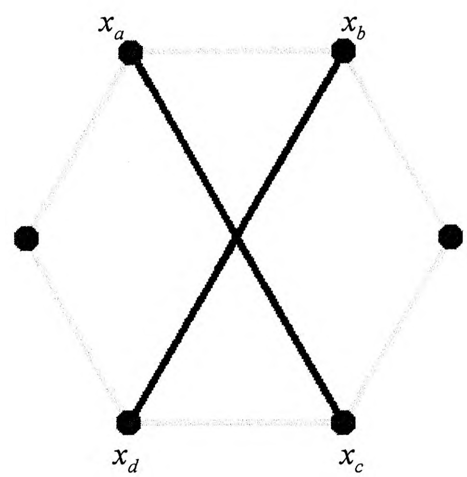

Fig. 5: An illustration of a cross

Case 2: Suppose that no crosses exist in G'. That is, if $a < b < c$, $b < d$, $x_a - x_c$, and $x_b - x_d$, then $d < c$. Now, consider a point x_p between x_a and x_b. Then there must exist x_q between x_c and x_d such that $x_p - x_q$. Similarly, if a point x_q exists between x_c and x_d, then there must exist x_p between x_a and x_b such that $x_p - x_q$. Hence, $x_a - x_c$ implies that $x_{a+1} - x_{c-1}$. Consider $l = \lceil \frac{a+c}{2} \rceil$. Then $\deg(x_l) = 2$. $\Rightarrow \Leftarrow$.

This is a contradiction because every vertex must have degree ≥ 3, and our original assumption (there do not exist any crosses in G') must be false. That is, there must exist at least one cross in G', which we can contract to a K_4 using the steps described in Case 1.

So, starting with graph H such that $\chi(H) = 4$, we can get an induced, color-critical sub-graph G. From here, we can contract G to G', and then contract G' to a K_4. That is, for any 4-colorable graph H, we can contract H to a K_4. ■

Chromatic Number of Non-planar Graphs

The second question raised from my research on the Four-Color Theorem is whether or not we can classify the chromatic number of non-planar graphs. To tackle this problem I am trying to develop a family of non-planar graphs with chromatic number $\leq k$. My first attempt at doing this is trying to add edges to a wheel. My initial questions are: If we start with a wheel, how many edges do we have to add until the graph is no longer planar? And does adding these edges change the chromatic number of the graph? To create a wheel, begin with a cycle of vertices. Then there is one additional point in the center of the cycle that is adjacent to every other vertex. The resulting graph is a wheel. Since wheels come in various sizes, we will tackle the proofs in two categories: even wheels and odd wheels.

New Theorems and Proofs: Even Wheel

First, we want to show that the wheel is a planar graph. In general, wheels consist of $k+1$ vertices and $2k$ edges. Also, since the smallest cycle is a three-cycle, with three vertices, then the smallest wheels must have at least four vertices. Suppose we have a wheel such that $v = k+1$ and $e = 2k$. Now, consider the inequality $e \leq 3v - 6$. That is, $2k \leq 3(k+1) - 6 = 3k - 3$. Then, $3 \leq k$. So, the inequality holds when $k \leq 3$. Substituting into $v = k+1$, we get that the inequality holds when $v \leq 4$, so it holds for any wheel since four is the minimum number of vertices in a wheel. Therefore, all wheels are planar.

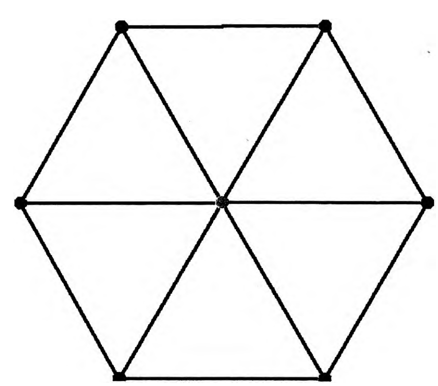

Fig. 6: An example of an even wheel

Consider an even wheel, *W*. The even wheel will look similar to the diagram (Fig. 6) which has six vertices one the outer cycle, however, *W* could have any even number of vertices on its outer cycle. To generalize the number of vertices, we say the even wheel has $2n$ vertices that make up the even cycle and one vertex in the middle. Therefore, *W* has $2n+1$ total vertices. We will denote this as $v = 2n+1$. Also, there are $2n$ edges incident with the center vertex, and $2n$ additional edges that make up the outer cycle. Thus, an even wheel with $2n+1$ vertices has $4n$ edges, denoted $e = 4n$.

Theorem 1: Let *W* be an even wheel. Then *W* is 3-colorable.

Proof: Let *W* be an even wheel. Since *W* contains an even cycle, we know the vertices that make up the cycle are 2-colorable. We will use red and blue for the cycle colors. Thus, there are *n* red vertices and *n* blue vertices, which alternate to form the outer cycle. The last vertex point in the wheel is the center point, connected to all the other vertices. Hence, it cannot be red or blue. We will use green to color the center vertex. Therefore, the even wheel is 3-colorable, as we have just described. ■

Theorem 2: Let *W* be an even wheel and let v_r be an arbitrary red vertex on the outer cycle of *W*. We can add $n-2$ edges incident with v_r without changing the chromatic number of *W*. That is, *W* will remain 3-colorable.

Proof: Let *W* be an even wheel and let v_r be an arbitrary red vertex on the outer cycle of *W*. We can add edges from v_r to every blue vertex, as well as the green center vertex without re-coloring any vertex. However, since v_r already connects to the green vertex, we cannot add that edge. Also, there are *n* blue vertices in the graph, but the red vertex already shares an edge with two of these vertices. Then, $n-2$ blue vertices remain which are not incident with

v_r and an edge can be drawn joining v_r to each of these $n-2$ vertices. This will add $n-2$ edges to W, while keeping the graph 3-colorable. ■

After adding $n-2$ edges to W, we want to know if it is possible for the graph to still be planar. That is, we want to find out if the graph satisfies Euler's formula, $v-e+f=2$. From Euler's formula we get the inequality $e \le 3v-6$. Thus, we want to see if W satisfies the inequality $e \le 3v-6$.

First, let W_1 be the new graph with the additional $n-2$ edges as described above. In W_1 we still have $v=2n+1$, but now $e=4n+(n-2)=5n-2$. Suppose the graph has at least 3 vertices, implying that $1 \le n$. To show this graph satisfies Euler's formula, we need to show that $e \le 3v-6$. That is, for this graph we need to show that $5n-2 \le 3(2n+1)-6=6n-3$. We will proceed by induction. For the base case, let $n=1$. Then $v=2(1)+1=3$ and $e=5(1)-2=3$. Since $3 \le 3(3)-6=3$, then inequality is satisfied when $n=1$. Now assume that $e \le 3v-6$ for some $n=h$, and we want to be sure that the inequality holds true for $n=h+1$.

We know from our assumption that:

$5h-2 \le 6h-3$

=> $-2+3 \le 6h-5h$

=> $1 \le h$

=> $1 \le h \le h+1$

Now, $h+1=6(h+1)-5(h+1)$ and $1=3-2$

=> $3-2 \le 6(h+1)-5(h+1)$

=> $5(h+1)-2 \le 6(h+1)-3$

That is, $5n-2 \le 6n-3$ when $n = h+1$. So, as long as the graph has at least three vertices, it will still satisfy Euler's formula after adding all possible edges incident to one red vertex as described above.

However, we are not limited to adding edges incident to only one of the red vertices. We can add edges incident to all of the red vertices in the same way as we added them to the one red vertex.

Theorem 3: Let W be an even wheel. If W_2 is a graph such that $n-2$ edges have been added incident to each of two arbitrary red vertices of W, then W_2 is 3-colorable but does not satisfy Euler's formula. That is, W_2 is non-planar and 3-colorable.

Proof: Consider graph W_2, where $n-2$ edges have been drawn incident to two of the red vertices. Hence, W_2 has $v = 2n+1$ and $e = 4n + 2(n-2) = 6n-2$. Then, $3v-6 = 3(2n+1)-6 = 6n-3$. We can clearly see that $6n-2 > 6n-3$, so the graph W_2 does not satisfy Euler's formula. Thus, it is non-planar, but the graph is still 3-colorable. ■

From Theorem 3, it follows that we can add up to $n(n-2)$ edges to W in this manner, with the same result.

Corollary 3.1: Let W be an even wheel. If W_n is a graph such that $n-2$ edges have been added incident to every red vertex of W, then W_n is 3-colorable but does not satisfy Euler's formula.

Proof: Let W be an even wheel. Then W has n red vertices. Now add $n-2$ edges incident to each of these n red vertices. Then, $e = 4n + n(n-2) \ge 4n + 2(n-2) = 6n-2 > 6n-3 = 3v-6$.

That is, $e > 3v - 6$. So, the graph W_n with the additional $n(n-2)$ edges does not satisfy Euler's formula. Therefore, W_n is non-planar and 3-colorable. ■

New Theorems and Proofs: Odd Wheel

We have already shown that a wheel is planar and multiple proofs on adding edges to an even wheel. Now, we will look at adding edges to an odd wheel, *L*. This wheel consists of an odd outer cycle with $2n+1$ vertices and has an additional vertex in the center of the cycle. Thus, *L* has $2n+2$ total vertices, denoted $v = 2n+2$. There are also $2n+1$ edges on the outer wheel as well as $2n+1$ edges incident with the center vertex. This gives us $4n+2$ total edges, denoted $e = 4n+2$.

Also, since the smallest odd cycle consists of three vertices, then the smallest odd wheel consists of four vertices. So, $4 \le v = 2n+2$ which means that $1 \le n$. However, consider an odd wheel where $v = 4$. Looking at the diagram (Fig. 7), we can see that this graph is a complete graph on four vertices, a K_4. As such, no additional edges can be added to this graph. Thus we cannot change the planarity or chromatic number by adding edges, and this graph is not particularly valid in this question. Therefore, we will only be looking at odd wheels such that $n > 1$.

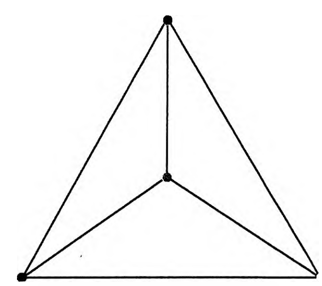

Fig. 7: An odd wheel with four vertices is a K_4.

Theorem 4: Let *L* be an odd wheel. Then *L* is 4-colorable.

Proof: Let *L* be an odd wheel. Since *L* contains an odd cycle, we know the vertices that make up the cycle are 3-colorable. We will use red and blue and yellow for the cycle colors. Thus, there are *n* red vertices and *n* blue vertices that alternate on the outer cycle as well as 1

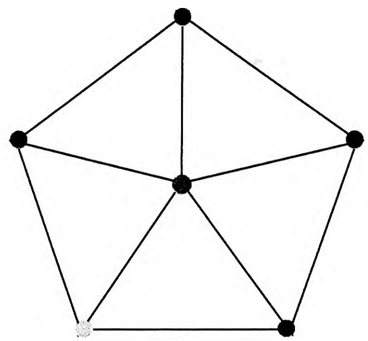

Fig. 8: An example of how to color the vertices of an odd wheel

yellow vertex, which all together form the outer cycle (Fig. 8). The last vertex point in the wheel is the center point, connected to all the other vertices. Hence, it cannot be red, blue, or yellow. We must use a fourth color, green, to color the center vertex. Therefore, the odd wheel is 4-colorable, as we have just described. ■

Theorem 5: Let L be an odd wheel and let v_y be the one yellow vertex on the outer cycle of L. Then we can add $2n-2$ edges incident with v_y, without re-coloring the vertices of L. That is, L remains 4-colorable.

Proof: Let L be an odd wheel and let v_y be the one yellow vertex on the outer cycle of L. Then v_y is adjacent to one red vertex, one blue vertex, and the green center vertex. Every other vertex is either red or blue, thus v_y can be connected to any of these other points without re-coloring the graph. Since there are $2n+2$ total vertices and we have accounted for 4 of them, we are left with $2n-2$ vertices which are not yet adjacent to v_y. Thus, we can add $2n-2$ edges connecting these vertices to v_y without re-coloring L. That is, L remains 4-colorable with the additional $2n-2$ edges as described. ■

Call this new graph L'. Similarly to the even wheel, we want to check the planarity of L' by seeing if it satisfies the inequality $e \le 3v-6$. For the graph L' we have $v = 2n+2$ and $e = 4n+2+(2n-2) = 6n$. Then, $3v-6 = 3(2n+2)-6 = 6n+6-6 = 6n$, and we have $e = 6n = 3v-6$. Clearly, this satisfies the inequality because they are equal, but adding any additional edges will make the graph non-planar. Now we want to see how many more edges we can add to L while keeping the graph 4-colorable.

Theorem 6: Let L' be an odd wheel with the additional edges described in Theorem 5 and let v_r be an arbitrary red vertex on the outer cycle of L'. We can add $n-2$ edges incident to v_r without changing the chromatic number of L. That is, L will remain 4-colorable.

Proof: Let L' be an odd wheel with the additional edges described in Theorem 5 and let v_r be an arbitrary red vertex on the outer cycle of L'. The vertex v_r can be incident to every blue vertex, as well as the one yellow vertex and the green center vertex without re-coloring any vertices of L. However, v_r already connects to the green vertex and the yellow vertex, so we cannot add those edges. Additionally, v_r is already adjacent to either 1 or 2 blue vertices.

Case 1: Assume v_r is adjacent to exactly one blue vertex. Then there are $n-1$ blue vertices which are not adjacent to v_r, and an edge can be drawn joining v_r to each of these $n-1$ vertices. This will add $n-1$ edges to L', while keeping the graph 4-colorable.

Case 2: Assume v_r is adjacent exactly two blue vertices. Then there are $n-2$ blue vertices which are not adjacent to v_r, and an edge can be drawn joining v_r to each of these $n-2$ vertices. This will add $n-2$ edges to L', while keeping the graph 4-colorable.

Therefore, in all cases L' remains 4-colorable. ■

Similar to the proofs for the even wheel, we are not limited to adding edges to just one red vertex. We can add these edges to every red vertex. It is necessary to note that only one vertex will satisfy Case 1 in the proof above. This will be the one red vertex that is adjacent to v_y in the original graph, L. Call this vertex v_1. The other $n-1$ red vertices will satisfy Case 2.

Theorem 7: Let L' be an odd wheel with the additional edges described in Theorem 5. Assume L'' is a graph such that $n-1$ edges have been added incident to v_1 and $n-2$ edges have been added incident to every other red vertex of L', as described above. Then L'' is 4-colorable but does not satisfy Euler's formula.

<u>Proof</u>: Let L' be an odd wheel with the additional edges described in Theorem 5. Now add $n-1$ edges incident to v_1 and $n-2$ edges incident to every other red vertex of L' as described in Theorem 6, and call this new graph L''. Since the red vertices are all joining blue vertices, it is not necessary to re-color the graph and so L'' remains 4-colorable. Now we need to check and make sure that L'' does not satisfy Euler's formula. First, we have that $e=(4n+2)+(2n-2)+(n-1)+(n-1)(n-2)=7n-1+n^2-3n+2=n^2+4n+1$ and $3v-6=3(2n+2)-6=6n$. That is, we know $e=n^2+4n+1$, $3v-6=6n$, and $n>1$. Now consider $e=n^2+4n+1>n^2+4n\geq 2n+4n=6n$. That is, $e>3v-6$ and does not satisfy the inequality $e\leq 3v-6$. Therefore, L'' is non-planar and 4-colorable. ∎

Conclusion

Although the proof of the Four-Color Theorem was published over twenty years ago, I find it extremely interesting that there is still no solution using basic math and logic. Since the Four-Color Theorem has led to the beginning of several mathematical subject areas and advancements in many other fields of mathematics, finding a solution of this type could possibly lead to innovative methods of proof to solve problems in these other areas. The proof I have done of Hadwiger's Conjecture for $p = 4$ could eventually lead me to a successful way to solve the conjecture for $p = 5$. As I noted earlier, the proof of Hadwiger's conjecture for $p = 5$ is equivalent to the Four-Color Theorem [6]. Thus proving Hadwiger's conjecture for $p = 5$ in a similar way to the proof of Hadwiger's conjecture for $p = 4$ may be a viable method to prove the Four-Color Theorem using logic skills instead of a computer program.

For the second part of my research I focused on a separate issue. Since we already know the maximum chromatic number of all planar graphs, it would be interesting to investigate similar results for various classes of non-planar graphs. While doing this, it will be valuable to show when non-planar graphs are four-colorable and at what point they cross over to a chromatic number greater than four. The proofs regarding the addition of edges to a wheel are the first steps toward defining and generalizing these different classes of non-planar graphs. They show how to modify even and odd wheels to make them non-planar and the maximum number of additional edges the graphs can have if they are to remain four-colorable. The next step in this area of research is to define additional types of graphs and explore their chromatic properties like we have done with the wheel. Ideally, it will be possible to categorize these various types of graphs into larger groups and classes of graphs with similar chromatic number properties.

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