Spring 4-29-2019

Mathematical Models: the Lanchester Equations and the Zombie Apocalypse

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The Analyzation of the Mathematical Models: The Lanchester Equations and the Zombie Apocalypse

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Senior Honors Project

Submitted in partial fulfillment of the graduation requirements of the Westover Honors College

Westover Honors College

April 2019

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Abstract

This research study used mathematical models to analyze and depicted specific battle situations and the outcomes of the zombie apocalypse. The original models that predicted warfare were the Lanchester models, while the zombie apocalypse models were fictional expansions upon mathematical models used to examine infectious diseases. In this paper, I analyzed and compared different mathematical models by examining each model's set of assumptions and the impact of the change in variables on the population classes. The purpose of this study was to understand the basics of the discrete dynamical systems and to determine the similarities between imaginary and realistic models. The Lanchester models that were examined were the Area Aimed model and the Aimed Fire model, while the Susceptible Zombie Removed model (SZR) and a Susceptible Infected Zombie Removed model (SIZR) portrayed the relationship between different population classes during the zombie apocalypse. From the differential equations used in the four models, I determined the impact of different variables on winning battles and on the likelihood of surviving the zombie apocalypse.

Keywords: Mathematical Models, Lanchester Equations, Square Law, Linear Law, Zombies

Introduction

I. Background and Literature Review

Mathematical modeling is a method of using systems, equations, and other tools to explain real world scenarios. A mathematical model is considered robust if the conclusions remain true even though the model may not be completely accurate (Meerschaert, 1993). Models are constantly being updated to increase their accuracy in representing reality and making better predictions.

There are different types of modeling, such as deterministic and stochastic models. Deterministic models include no random factors and always produce the same output from a given starting point (Lawson and Marion, 2008), while models that include randomness and predict the distribution of various possible outcomes are known as stochastic models.

Mathematical models are also divided considering the time extent of the model. The continuous process describes systems that are constantly changing, while the discrete process describes systems in which the bulk of the activity occurs in a short period of time (Mooney and Swift, 1999). Dynamical system models are the most common type of dynamic model and uses differential equations to represent the forces of change (Meerschaert, 1993). Dynamic systems are known for being easy to formulate, but difficult to solve. This is because dynamic systems seldomly use linear differential equations and therefore possess a nonlinear relationship to depict the influence of different variables on various classes. Models in which the change in behavior uses the discrete process and are deterministic are called discrete dynamical systems (Mooney and Swift, 1999), which is the form of the mathematical models analyzed within this study. Discrete dynamical systems are the best for the research study because the change in the different population classes occurs during a short period of time, which makes it easier to analyze.

In this study, I examined two general types of models. The first set of models analyzed were the Lanchester models, which were renowned for their simplicity and used to predict war outcomes. The second type of mathematical models depicted the fictional outcomes of a possible zombie apocalypse. Although both types of models were discrete dynamical systems, the models differed greatly in regards to their background, process for solving, and the structure of the respective differential equations.

Lanchester Models

The Lanchester models are systems of differential equations used to describe aspects of conventional warfare, which are wars without the use of chemical, biological, or nuclear weaponry (Epstein, 1985). These models are renowned for their simplicity and are often featured in introductory courses of mathematical modeling (Mackay, 2005).

The Lanchester models were derived from Lanchester's famous laws, which were simple to understand and eased the transition into more complex topics of mathematical modeling. His most renowned equation was the Square Law which predicted outcomes of battles within the Aimed Fire model. The other lesser known Lanchester equation examined was the Linear Law, which was used within the Area Aimed model.

These models had vast amounts of research and publications which analyzed the Lanchester equations and explained the background, applications, and flaws within the models in an understandable manner. Their simplicity allowed for hand calculations, which provided me the opportunity to understand the relationship between the variables using written methods rather than the use of computer programming.

Mathematical modeling was first used to explain conventional warfare by F. W. Lanchester in 1915. Frederwick William Lanchester was a well-known engineer born in Lewisham, England in 1868 (Ricardo, 1948). He was a man of many talents and made contributions to various fields, most notably his advancements within the automobile industry. In his book, *Aircraft in Warfare* (1916), he introduced two sets of differential equations to explain aspects of aerial warfare, which were later adapted into models for various battle situations (Epstein, 1985). The Lanchester models predicted the winner, duration of the war, and the survivors at time (*t*) for a specific battle.

The differential equations used in the both the Area Aimed and Aimed Fire models described the outcomes of two battling armies as functions of time and were dependent on the relative attacker and defender strengths. The relative strength of an army was contingent upon which model was used, the Aimed Fire model or the Area Aimed model. For the Lanchester models to accurately describe the outcomes of a battle, the models required specific assumptions to be met (Dolansky, 1962). The assumptions required for each model limited the type of battle which the Lanchester models could accurately describe. Listed below are the general assumptions used in both the Lanchester models.

Assumptions:

- No possibility of withdrawal
- No significant differences between the offensive and defensive armies (i.e. two homogenous forces were engaged in combat)
- The two forces were within weapon range of the opposing army
- The rate at which the armies were reduced (the attrition rate) was known and held constant throughout the duration of the battle.

To examine the Lanchester models, I supposed there was a battle between two forces, labeled as the *X men* and *Y men*, and that all the assumptions for the general Lanchester models were met. The variables used in both the Aimed Fire model and Area Aimed model were labeled below. The fighting efficiency coefficients were numerical values used to represent the respective army's ability to fight, which included the average skills of the soldiers and resources available as well as other various factors (Epstein, 1985).

Variables:

x = the fighting efficiency of the *X men*

y = the fighting efficiency of the *Y men*

X_t = the number of *X men* units at time t

Y_t = the number of *Y men* units at time t

Aimed Fire Model

The Aimed Fire model described battle scenarios in which the exact location of the enemy was known, which allowed soldiers the ability to immediately shift their fire to a new target after their previous target had been eliminated.

The Aimed Fire model used the Lanchester's most renowned law, the Square Law, to depict the outcomes of a battle. Within the Aimed Fire model, an army's fighting strength was equivalent to the enemy's fighting efficiency coefficient times the square of the number of enemy units.

$$\frac{dX}{dt} = -yY_t \qquad \frac{dY}{dt} = -xX_t$$

The Square Law described the change in an army to be negative of the opposing army's fighting strength. To obtain a single differential equation that calculated the change of both armies, I combined the two equations above and integrated.

(1)

$$\frac{\mathrm{dY}}{\mathrm{dX}} = \frac{-\,xX_t}{-\,\mathrm{y}\mathrm{Y}_t}$$

$$yY_t\,\mathrm{dY} \;=\; x\mathrm{X}_\mathrm{t}\,\mathrm{dX}$$

$$\int yY_t = \int x\,X_t$$

$$yY_t^2 \;-\; xX_t^2 \;=\; constant\ (C)$$

Hence, $yY_t^2 \;-\; xX_t^2 = \mathrm{C}$ was the Lanchester's Square Law, which was used to determine the outcomes of the aimed fire model.

<u>Area Aimed Model</u>

The next Lanchester model I examined was the Area Aimed model which used the Linear Law to explain warfare. The Area Aimed model described battle scenarios in which the exact location of each army was undisclosed. Therefore, armies were aware of only the general location of their enemies and lacked the immediate feedback featured within the Aimed Fire model (Dolansky, 1962). The Area Aimed model assumed that the fire from the surviving units would be distributed uniformly over the general area rather than focused on a specific target. The change of an army was described as the negative fighting efficiency coefficient of the opposing side times both the army's number of forces, which was illustrated in the differential equations below (Lepingwell, 1987).

$$\frac{dX}{\mathrm{dt}} = -\mathrm{X}_\mathrm{t}y\mathrm{Y}_t \qquad \frac{\mathrm{dY}}{\mathrm{dt}} = -\mathrm{Y}_t xX_t$$

The Lanchester's Linear Law was achieved by the combination of the two original differential equations to include both variables for the *X men* and *Y men*.

(2)

$$\frac{dY}{dX} = \frac{-Y_t x X_t}{-X_t y Y_t}$$

$$X_t y \int Y_t \, dY = Y_t x \int X_t \, dX$$

$$X_t y({Y_t}^2) = Y_t x({X_t}^2)$$

$$yY_t = xX_t$$

$$yY_t - xX_t = \text{constant (C)}$$

Hence, $yY_t - xX_t = C$ was the Lanchester's Linear Law, which was used to determine the winner, the losses, and duration of the war for the Area Aimed model. The Lanchester models depicted the real-life situation of warfare, but differential equations were also be used to predict outcomes of fictional situations such as the zombie apocalypse.

Zombies

The inspiration for the fictional monster were derived from cultural beliefs and myths in the vodou religion (Morgan, 2016). The original stories described a powerful vodou sorcerer, known as a Bokor, who brought people back to life to serve as their personal slaves. Becoming a zombie was viewed a form of punishment for wronging the Bokor during their time alive. The word 'zombie' originated from the Haitian creole word meaning spirit of the dead (Morgan, 2016). In the original folklore, zombies were depicted as mindless, reanimated corpses and were tame compared to modern adaptations. The original zombies were submissive and usually didn't attack people unless it was a Bokor's command. They also typically responded in groans or moans and were sluggish and clumsy when they moved.

The process of becoming a zombie occurred through use of spells and magic powder. After taking the magic powder, the victim's heart and breathing would become suppressed which caused their body temperature to drop to the extent that they appeared dead. Once the victim was pronounced dead, the victim was buried and later exhumed by the Bokor. The victims were only able to understand basic commands and had superior strength since they lacked basic responses to stimuli, which made them unable to feel pain or exhaustion.

The folklore regarding zombies included the possibility of a cure to the Bokor's curse. In certain myths, a zombie's soul would be contained within a vessel and returned to its' body if the vessel was broken. Another myth stated that if zombies were fed salt, they would regain their senses, but would remain in a diminished mental state and be vulnerable to recapture. Some also believed that a victim would regain their freedom if the Bokor died. In all these possibilities, the victim never fully returned to their prior state, unless they were cured through a divine intervention.

The myths of the Bokor's curse have given zombies their undead quality. Undead zombies supported the representation of the removed class as depicted by the differential equations for the models of the zombie apocalypse used in the study. The removed class included individuals who were removed from the general population, but unlike other removed groups in general mathematical models, zombies were resurrected and therefore able to transition from the removed group to the zombie population.

Zombies were first introduced to the English language by Robert Southey's 1819 novel, "A History of Brazil", but became popular phenomenon through the film industry (Morgan, 2016). Early film adaptations of zombies strongly resembled the victims described in the Haitian folklore, but also gave them unique weaknesses such as fear of fire and bright lights. The 1959 film, *Teenage Zombies*, was the first introduction of the zombie as a form of disease rather than a vodou curse. The film described the origins of zombies by being exposed to nerve gas, which inspired the common notion of zombies as a deadly infectious disease.

George A. Romero has been credited with the creation of the modern zombie. Romero's depiction of the zombie apocalypse was very popular and known for its excessive amounts of gore. The films first introduced the transference of the disease through bite, which allowed the modern zombies to quickly outnumber the humans. The rate at which the disease was transmitted was a variable used to describe the interaction between population classes within the zombie mathematical models.

Romero also initiated the classic method in stopping zombies by destroy their brain. Popular culture has expanded Romero's original method of killing zombies to include decapitation, bludgeoning, burning, and explosions (Henriksen, 2004). The death rate of zombies was depicted in the zombie models and described the transition of zombies to the removed class. The zombies depicted in the films were slow moving, undead, and aggressive towards humans. Zombies were depicted as only craving human flesh, which was an assumption for the Susceptible Zombie Removed and Susceptible Infected Zombie Removed model, therefore the existence of zombies did not affect other life form populations.

The zombies analyzed in this study were based upon the zombies popularized in George A. Romero's films. His movies introduced crucial concepts involving zombies such as the idea of a removed class, method of transmitting the virus, and ways of killing zombies which were variables included within the zombie mathematical models to describe the transitions between the class populations.

Zombie Model

There have been numerous publications regarding mathematical models to depict the different scenarios of the zombie apocalypse. These mathematical models were inspired by previous models for other infectious diseases such as the bubonic plague (Smith, 2014). The models used in this study were created through the collaboration of Philip Munz, Ioan Hudea, Joe Imad, and Robert Smith (2009). Within their work, the authors included various models evaluating the relationship between zombies and susceptible humans, but I particularly studied the Susceptible Zombie Removed and Susceptible Infected Zombie Removed model.

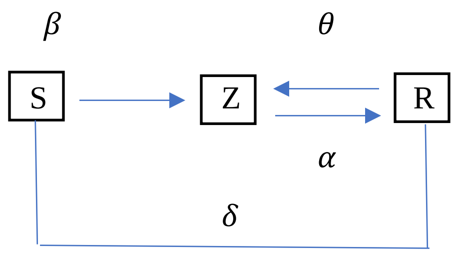
Unlike the Lanchester models, which represented a realistic battle situation, the zombie models were inspired by a science fiction phenomenon. The zombies used within these models were assumed to be slow moving, undead cannibals, who only desire human flesh and therefore did not affect any other life form's population.

Susceptible Zombie Removed (SZR) Model

The basic model within the study conducted by Munz et al. was known as the Susceptible Zombie Removed (SZR) and divided the general population into 3 classes. The susceptible class were identified as healthy individuals that were vulnerable to contracting the disease. At the start of the model, most of the population originated as susceptible humans. Next, was the zombie category, which defined humans who had been resurrected from the dead after contracting the zombie disease. Lastly, were those removed from the population which occurred either by being attacked or by natural causes (i.e. not zombie related). Unlike other mathematical models, individuals were able to be leave the removed group by being resurrected as a zombie. Zombies were defined as the undead and therefore for a susceptible human to become a zombie they must have contracted the disease, died, then arose as a zombie. The different variables and relationships between the three class of individuals were depicted and listed in equation 3.

(3)

II



$dS = II - \beta SZ - \delta S$

$dZ = \beta SZ + \theta R - \alpha SZ$

$dR = \delta S + \alpha SZ - \theta R$

<u>Variables</u>

S_t – Healthy Humans at time (t)

Z_t – Zombies at time (t)

R_t – Removed at time (t)

δ – Death rate of susceptible humans by natural causes (i.e. non-zombie related)

β – Transmission rate

α – Death rate of zombies (caused by destroying its brain or removing its head)

θ – Resurrection rate (susceptible to zombie)

II – Birthrate

βSZ – Transmission of infection between Susceptible and Zombie

δS – Susceptible who died of natural causes

θR – Resurrected dead who became Zombie

αSZ – Zombie who were disposed of by Susceptible

Assumptions

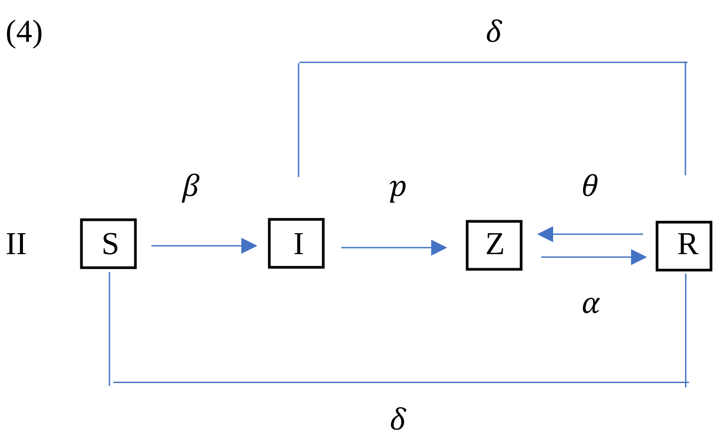
- Humans were the only life forms considered within the model
- Birthrate was held constant
- Zombies did not attack other zombies
- A susceptible immediately died after contracting the disease

The variables relationship between the 3 classes are depicted in the diagram and equations above (3). The SZR model assumed that the zombie virus immediately killed infected humans. Hence, the model did not allow for the possibility of an infected carrier.

Susceptible Infected Zombie Removed (SIZR) Model

The next model, SIZR, added another population class and the possibility of an infected carrier, labeled as Infected. The infection period began after a human was infected with the zombie disease and ended when the infected individual died. During the infection period, the infected could have died of natural causes or infected other susceptible humans. The model included the general variables and assumptions as listed in equation 3 as well as a few additional assumptions listed below

(4)



dS = II - $\beta SZ - \delta S$

dI = $\beta SZ - pI - \delta I$

dZ = $pI + \theta R - \alpha SZ$

dR = $\delta S + \delta I + \alpha SZ - \theta R$

Variables:

I – Infected humans

p – rate of infection

pI – transfer of Infected to Zombie

δI – natural death of Infected

Assumptions:

- After a susceptible human is infected with the zombie virus, the susceptible becomes a member of the Infected class

The variables relationship between the 4 classes were depicted in the diagram and equations above (4). The SIZR model created a new population class and allowed for the possibility of an infected carrier.

II. Purpose

The purpose of this paper was to understand and examine the differences between different types of mathematical models. Each type of model had its own variables, assumptions, and methods of solving. The study illustrated that mathematical models could be used to represent fictional situations as well as realistic. It also demonstrated the development of mathematical modeling from the simple equations used in the Lanchester models to the complex relationships between the different populations of the zombie apocalypse. The Lanchester models were created when aircraft warfare was becoming popular and were based upon simple differential equations that could be solved by hand. The zombie models were more recent and used complex differential equations that required computer programing to solve. I examined and compared a basic and complex model of each general type of model, then compared the overall themes of the Lanchester and zombie models. The Area Aimed model and the SZR model were the basic models, respectively, while the Aimed Fire model and the SIZR model were complex. The more complex models included additional variables which allowed for more modeling possibilities amongst the variables.

Research Question and Hypothesis

Suppose there was a battle situation such that one side could convert soldiers to fight for the opposing side. Instead of the two forces being mutually exclusive, as illustrated in the Lanchester models, the soldiers would transition between the classes showcased in the zombie models. Rather than only being categorized alive or dead, the soldiers had the possibility of becoming the enemy that they had fought against originally, which would be the case when modeling a zombie apocalypse. It might become possible for these battle situations to become a reality depending upon the advancement of brainwashing and manipulation or even the possibility of a zombie apocalypse.

I hypothesized that there will be multiple similarities between the Lanchester and zombie models because of the use of differential equations to explain the relationships between classes in discrete dynamical systems. After completing the study, I gained a better understanding of the diversity in mathematical modeling systems and the differential equations used within the models.

Methods

There was a total of 4 models which were analyzed and cross referenced within the study. Each model was analyzed individually by examining its assumptions and variables in relation to the different classes. The results derived from the models were then compared to its sister model to determine which variables yielded the greater effect for each model. Therefore, the Aimed Fire model and Area Aimed model were first analyzed individually and then compared to determine the impact of the fighting efficiency coefficient and the number of forces on the winner, the losses, and time of a battle. For the zombie models, the SZR and SIZR model were individually examined, then compared to analyze the various variables impact of the different classes' populations. Finally, the general aspects of the Lanchester models were compared to the zombie models, which included the subject matter, specificity of assumptions, method of solving, and type of differential equations used.

To understand the implications of the Lanchester models, I solved the equations using random variables and replicated battle situations in which total annihilation for both the *X men* and *Y men* occurred. The equations required properties of differential equations and Laplace transformations using hyperbolic functions of sine and cosine. To analyze the Lanchester models, I assumed the initial number of *Y men* was greater than initial number *X men*, such that the initial number of *Y men* equaled a real number times the number of *X men*. I wanted to determine the numerical value for the fighting efficiency coefficient for both the *X men* and *Y men* for *X men* to win.

During my background research for the Lanchester models, I discovered the Munz et al. (2009) article that used mathematical modeling to determine the outcomes of the zombie apocalypse. I was very interested and wanted to investigate more comprehensive models that depicted situations that had not occurred. These equations remained discrete dynamical equations, similar to those used in the Lanchester models, but included a greater number of variables and population classes. To solve the complex equations, it required the use of the mathematical computer software program Maple. Maple allowed for easy analyzation, visualizations, and solutions to the specific differential equations. The model depicted a doomsday scenario in which susceptible humans and zombies could not coexist. Hence, either zombies did not exist or if the zombie virus was real, then eventually all susceptible humans would be dead or turned. To analyze the variables, I ran a sensitivity analysis to determine the effect of each variable in both the SZR and SIZR model. Maple produced graphs that were used to make visual conclusions and depicted the relationship between the different classes in yearly time increments.

Analysis and Results

For the Lanchester models, I assumed there was a battle situation in which the *Y men* had a greater amount of forces. The goal was to determine the numerical values for the fighting efficiency for both the *Y men* and the *X men* for the *X men* to win. For the Aimed Fire model, if the initial amount of *Y men* was *n* times the amount of *X men*, then the fighting efficiency of the *X men* $>n^2$ fighting efficiency of y. The statement was proved by solving for total annihilation which occurred when both the *X men* and *Y men* forces equaled 0.

Let $Y_0 = nX_0$ for some real number *n* and let the fighting efficiency of the *X men*, x, equal ay for some real number *a*. Solving for *a* such that both the *Y men* and *X men* forces are annihilated, that is $yY^2 - xX^2 = 0$ and gives the following

(5) $$y(nX_0)^2 - ay(X_0)^2 = 0$$

$$n^2yX_0^2 - ayX_0^2 = 0$$

$$n^2yX_0^2 = ayX_0^2$$

$$n^2 = a$$

Therefore, for total annihilation to occur the fighting efficiency of the *X men* = n^2y.

If $Y_0 = nX_0$ and the fighting efficiency for the *X men* > n^2y, then the *X men* win. While, if $Y_0 = nX_0$ and the fighting efficiency for the *X men* < n^2y, then the *Y men* win. Also, for the *X men* to win if $Y_0 = nX_0$, the fighting efficiency of the *Y men* < $\frac{1}{n^2}$ x. This was proven by solving for total annihilation and letting $Y_0 = nX_0$ for some real number *n* and let the fighting efficiency of the *Y men*, y, equal *a*x for some real number *a*. Solving for *a* such that both the *Y men* and *X men* forces are annihilated gives:

(6) $$yY^2 - xX^2 = 0$$

$$ax(nX_0)^2 - x(X_0)^2 = 0$$

$$an^2xX_0^2 - xX_0^2 = 0$$

$$an^2xX_0^2 = xX_0^2$$

$$an^2 = 1$$

$$a = \frac{1}{n^2}$$

Therefore, for total annihilation to occur the fighting efficiency of the *Y men* = $\frac{1}{n^2}x$. If $Y_0 = nX_0$ and the fighting efficiency for the *Y men* > $\frac{1}{n^2}$ fighting efficiency of x, then the *Y men* win. While, if $Y_0 = nX_0$ and the fighting efficiency for the *Y men* < $\frac{1}{n^2}$ fighting efficiency of x, then the *X men* win.

For the Area Aimed model, I found that if $Y_0 = nX_0$, then the *X men* fighting efficiency $>n$ times the fighting efficiency of y. This is proven by solving for total annihilation, that is for when *X men* and *Y men* forces = 0. Let $Y_0 = nX_0$ for some real number n and let the fighting efficiency of the *X men*, x, equal *a*y for some real number *a*. Solving for *a* such that both the *Y men* and *X men* forces are annihilated gives:

(7) $$yY - xX = 0.$$

$$y(nX_0)^2 - ay(X_0)^2 = 0$$

$$nyX_0 - ayX_0 = 0$$

$$nyX_0 = ayX_0$$

$$n = a$$

Therefore, for total annihilation to occur the fighting efficiency of the *X men* must be $= nx$. If $Y_0 = nX_0$ and the fighting efficiency for the *X men* $> n$ time fighting efficiency of y, then the *X men* win. While, if $Y_0 = nX_0$ and the fighting efficiency for the *Y men* $< \frac{1}{n}$ times the fighting efficiency of x, then the *Y men* win.

Also, if $Y_0 = nX_0$, then the *Y men* fighting efficiency $< \frac{1}{n}$ times the fighting efficiency of x for the *X men* to win. This was proven by solving for total annihilation, and letting $Y_0 = nX_0$ for some real number *n* and let the fighting efficiency of the *Y men* be y = *a*x for some real number *a*. Solving for *a* such that both the *Y men* and *X men* forces are annihilated gives:

(8)

$$yY - xX = 0.$$

$$ax(nX_0) - x(X_0) = 0$$

$$anxX_0 - xX_0 = 0$$

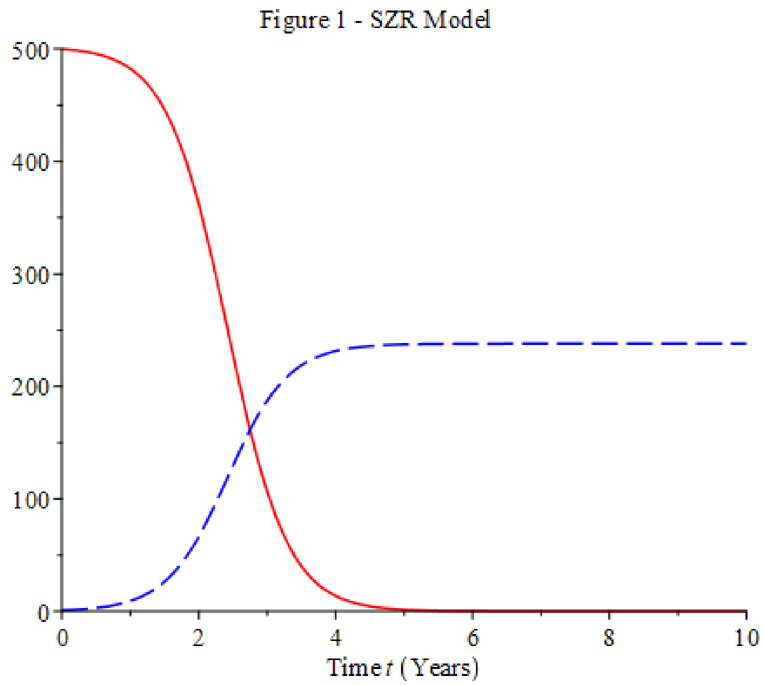
$$anxX_0 = xX_0$$

$$an = 1$$

$$a = \frac{1}{n}$$

Therefore, for total annihilation to occur the fighting efficiency of the *Y men* must be $= \frac{1}{n}x$. If $Y_0 = nX_0$ and the fighting efficiency for the *Y men* $> \frac{1}{n}$ times the fighting efficiency of x, then the *Y men* win. While, if $Y_0 = nX_0$ and the fighting efficiency for the *Y men* $< \frac{1}{n}$ times the fighting efficiency of x, then the *X men* win.

The Lanchester models were simple enough to be able to solve for the solutions by hand, but to interpret the zombie models, SZR and SIZR, I used the mathematical modeling software, Maple. I began by using variables as described in the original study which were achieved by using MATLAB to solve the equations using Euler's method (Muniz et al., 2009). The variables for the SZR model were II = 0, $\alpha = .005, \beta = .0095,\ \delta = 0,$ and $\theta = .0001$. Figure 1 illustrates the outcomes of the SZR model using the standard variables. The solid red line represented the susceptible human population, while the dashed blue line represented the zombie class.



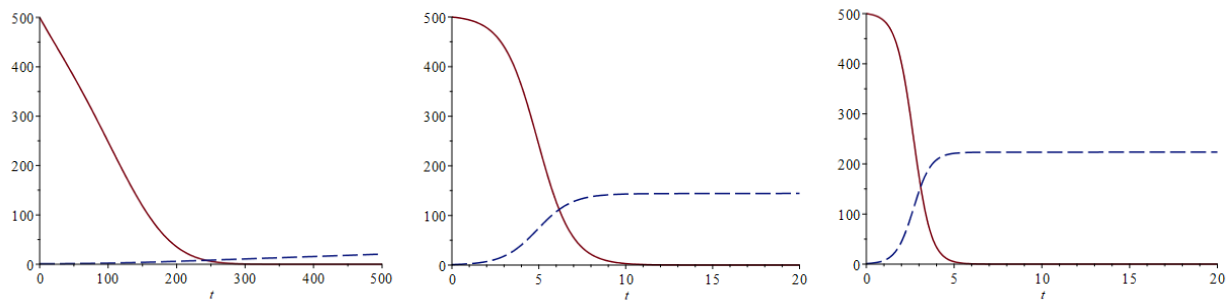
To understand the effects of each variable on the outcome, I ran a sensitivity analysis. A sensitivity analysis is the study of the effects of changing different variables on the outcome of a mathematical model (Mooney and Swift, 1999). Therefore, each variable was assigned different values, while the others remained constant to analyze the specific variable's effect upon the trends of the different population classes. The range of different decimal placements for each variable made it not possible to use the same numerical values for each variable, which would rarely be the case. Maple produced graphs for each value, which was then examined to determine the change in the classes. For the SZR model, I analyzed the time at which the total susceptible human population was annihilated and the greatest number of zombies.

The birthrate variable, II, specifically pertained to the susceptible population. An increase in the birthrate did not present an effect upon the outcomes of zombie apocalypse. As demonstrated in Figure A of the appendix, regardless of the numerical value assigned to II, the susceptible population was eradicated before 5 years and the numbers of zombies stabilized around 150 zombies. Therefore, increasing the background birthrate of humans did not affect the outcomes of the zombie apocalypse in the SZR model.

Unlike II, the transmission rate, β, greatly affected the class trends as exemplified by the graphs from the sensitivity analysis in figure B.

Figure B - Sensitivity Analysis of β using SZR model

β = 0.005 at t = 500 β = 0.007 at t = 20 β = 0.009 at t = 20



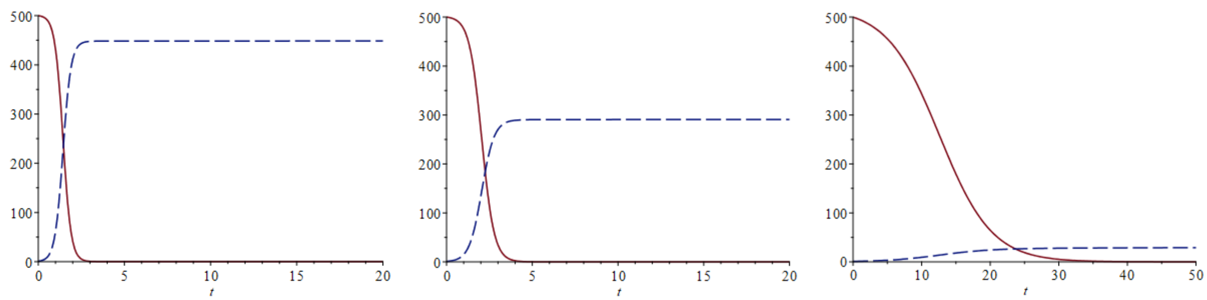
If β increased, then the time at which the susceptible human classes was eradicated decreased. The time for annihilation drastically decreased from about 250 years for β = 0.005 to about 5 years for β = 0.009. An increase in β also resulted in an increase in the maximum number of zombies. An altercation between zombies and humans ends with either a zombie killed or a human infected. Therefore, increasing the rate of transmission, while the rate at which zombies were killed was kept constant, would result with an increase in the overall number of zombies and a decrease in the time until human extermination.

If the death rate of non-zombie instances, δ, increased, then the maximum number of zombies decreased as portrayed in appendix figure C. An increase in the death rate meant that there was an increase in dead susceptible humans. Since these individuals did not contract the zombie disease they were not resurrected as zombies, which explained the decrease in the maximum number of zombies from about 200 for δ = 0.1 to about 20 for δ = 1. A change in the death rate also caused slight inconsistencies with the time at which the susceptible humans were annihilated but remained roughly around 5 years. Therefore, an increase in the rate of transmission would cause a decrease in the overall number of zombies, while the time for human extermination remained relatively constant.

Unlike the other variables in the SZR model, the death rate of zombies, α, was able to delay the annihilation of humans and decrease the number of zombies as portrayed in figure D.

Figure D – Sensitivity Analysis of α using SZR model

α = 0.001 at t = 20 α = 0.004 at t = 20 α = 0.009 at t = 50

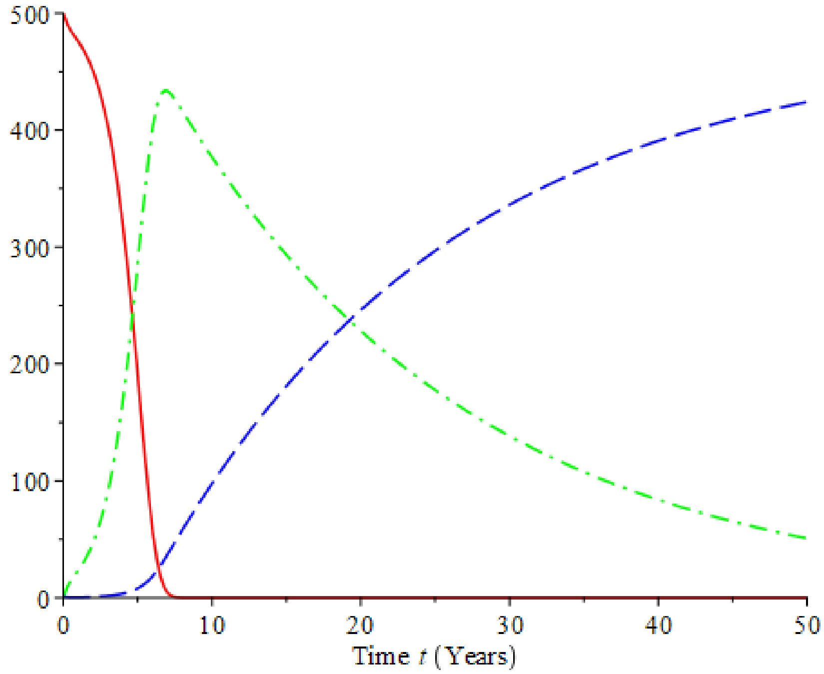


The variable α depicted the probability that a susceptible human was able to kill a zombie. An increase in α caused more zombies killed, which result in fewer zombies in circulation and therefore would require more time to eradicate the susceptible human class. Despite being able to prolong the zombie apocalypse, increasing α would not allow the susceptible class to be able to annihilate the zombie population.

An increase in the resurrection rate, $\theta,$ resulted a decrease in the time needed for susceptible class extermination and increase in number of zombies as illustrated in appendix figure E. The resurrection rate was the rate at which the zombies arose from the dead, therefore if θ increased, the the number of resurrected zombies increased, which in turn increased the total number of zombie in existence. The change in resurrection rate caused the time for human extinction to decrease, but not as drastically as the rate of transmission. For θ = 0.1 the time for extinction was a little above 4 years. While for θ = 1, the time for extinction was about 3 ½ years. The change in β had a time difference of 245 years, while the change in θ only resulted in a difference of half a year.

The SIZR model added another population class and allowed for the possibility of an infection carrier with the ability to transmit the disease to susceptible humans. Infected carriers either died before the zombie disease transmission was completed or were resurrected as zombies. Similarly to the SZR model, I used the variables from the original study, which were II = 0, $\alpha = .005, \beta = .095,\ \delta = 0.0001, \theta = .001,$ and $p = .05$ (Munz et al., 2009). Figure 2 illustrated the outcomes of the SIZR model using standard the variables with the susceptible population class depicted by the red solid line, zombie class represented as the dashed blue line and infected class symbolized with the alternating green alternating dotted dash line.

Figure 2 - SIZR Model



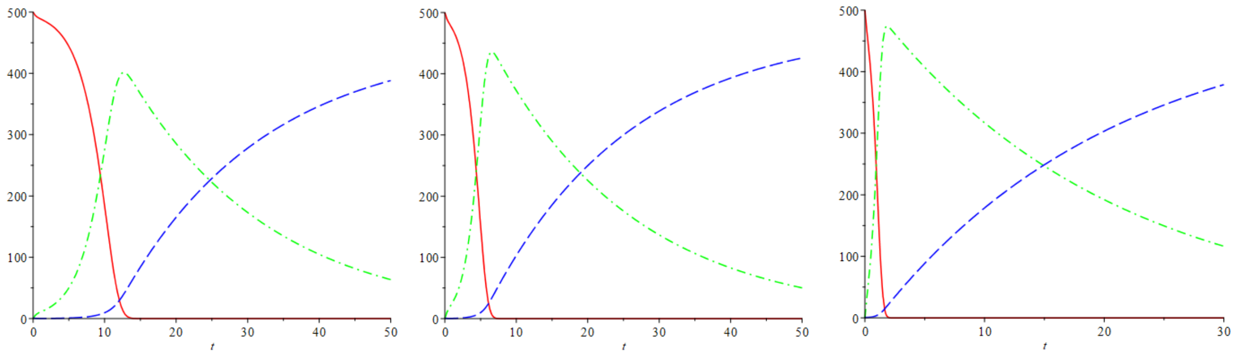
The SIZR model yielded similar results to the SZR model with the addition of another variable and population class. To analyze the sensitivity analysis, I examined the time for human annihilation, the maximum number of infected, and the time at which the zombie class surpassed the infected class.

The change in birthrate, II, did not affect the trends of the population classes as portrayed by figure F within the appendix. The changes in the birthrate was consistent between the two zombie mathematical models, because in both the SZR and SIZR models, II did not affect the population trends.

The sensitivity analysis of the transmission rate of the zombie disease, β, is illustrated in figure G.

Figure G – Sensitivity Analysis of β using SIZR model

β = 0.05 at t = 50 β = 0.1 at t = 50 β = 0.5 at t = 30



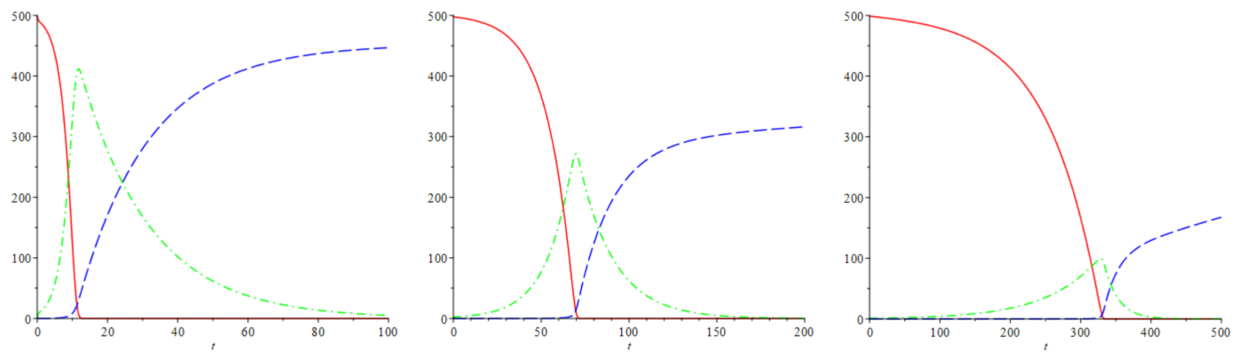
An increase in β resulted in a decrease in time till human extinction from about 14 years at β = 0.05 to about 2 years for β = 0.5. It also produced an increase in the maximum number of infected and decreased the time at which the zombie class surpassed the infected. β represented the rate of transmission, therefore an increase in β meant that there was an increase in number of individuals infected, hence the increase in maximum number of infected. Since the rate of infection remained constant, while the number of individuals infected continued to increase, it caused more zombies to arise which resulted in the decrease in the time need for the zombie class to become the majority.

The increase of the death rate of non-zombie related instances, δ, had a general decrease amongst the population classes. An increase in the death rate of susceptible humans decreased the time of human extinction, because more susceptible humans were dying by two methods, zombie and non-zombie related deaths. It also decreased the number of infected, because there were fewer susceptible humans to infect. Since the death were non-zombie related and there were less humans being infected, there would also be a decrease in the zombie class. There was also a decrease in the time at which the zombie class surpassed the infected, which was because there were fewer humans being infected. Since the resurrection and infection rate remained constant the infected class decreased further while the zombie class increased. The change in δ was illustrated in the appendix by figure H.

Similar to the results of the SZR model, the death rate of zombies, α, was the only variable that delayed human extinction. As the death rate of zombies increased, the number of zombies and infected decreased, while the time for human extinction and time for zombies to exceed the infected class increased. The death rate's effect on the population classes was demonstrated in figure I below.

Figure I – Sensitivity Analysis of α using SIZR model

α = 0.01 at t = 100 α = 0.04 at t = 200 α = 0.08 at t = 500



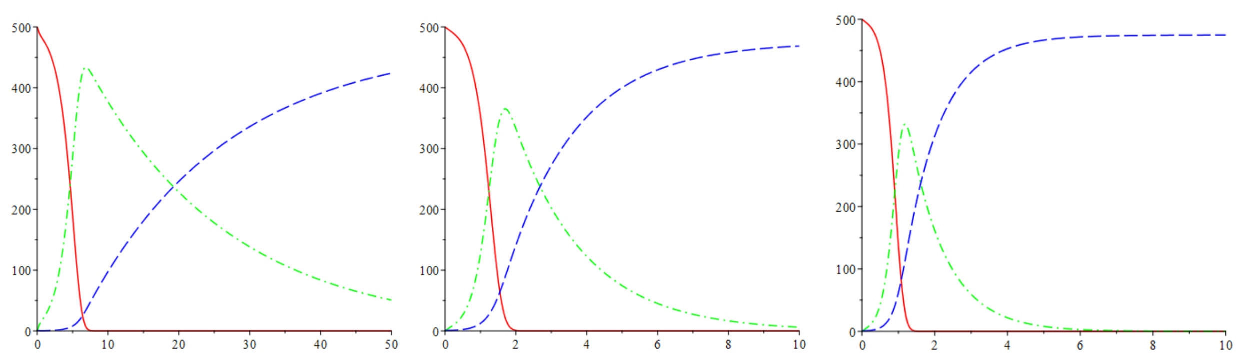
An increase in the death rate of zombies implied that there was an increase in the number of interactions between zombie and humans in which the zombie was killed rather than the human being infected. Therefore, the number of zombies and infected population decreased and the time increased.

The effects of increasing the resurrection rate, θ, resulted in different outcomes than the SZR model. In the SIZR model, an increase in the θ did not affect the trends of the population trends as illustrated in the appendix by figure J. By introducing a new variable to represent the rate of infection in the SIZR model, the new variable decreased the significance of the resurrection rate. In the SZR model, θ had a slight impact, therefore any diminution of θ's function would result in a smaller effect upon the outcomes.

The SIZR model introduced a new variable, the rate of infection *(p)*, as well as a new population class. An increase in *p* yielded a decrease in the time for human annihilation and the time for zombie class to surpass infected class. It also resulted in a decrease in the maximum number of infected. The effect of the rate of infection on the population trends were demonstrated by figure K.

Figure K – Sensitivity Analysis of *p* using SIZR model

p = 0.05 at t = 50 | *p* = 0.5 at t = 10 | *p* = 1 at t = 10



Increasing the rate of infection meant a decrease in the infection period. Therefore, the zombie disease took less time to kill its victims, which in turn created more zombies and decrease the maximum number of infected. Since there were more zombies in existence, it decreased the time to eradicate humanity.

Table 1 is a comparison table that detailed the outcomes of the sensitivity analysis for each variable in the SZR and SIZR model. Both the SZR and SIZR model examined the time at which the susceptible human population equaled 0, which was represented by “t for Susceptible = 0”. The SZR model also examined the total zombie population which was represented in the table as “# Zombies”. For the SIZR model, I examined the maximum number of infected, represented as “max Infected”, and the time at which the zombie population became the majority population class excluding the removed class. The time for the zombie population to reach the majority was represented in the table as “t for Zombie +”.

Table 1. Sensitivity Analysis Comparison for SZR and SIZR models

SZR	**SIZR**
II – constant	II – constant
β – decrease t for Susceptible = 0 increase # Zombies	β – decrease t for Susceptible = 0 increase max Infected decrease t for Zombie +
δ – t for Susceptible = 0 inconsistent decrease # Zombies	δ – decrease t for Susceptible = 0 decrease max Infected decrease t for Zombie +
α – increase t for Susceptible = 0 decrease # Zombies	α – increase t for S = 0 decreased max infected increased t for Zombie +
θ – decrease t for Susceptible = 0 increase # Zombies	θ – constant
	p – decrease t for Susceptible = 0 decrease max Infected decrease t for Zombie +

There were many similarities within the sensitivity analysis of the SZR and SIZR models. In both models the birthrate was constant and the rate at which zombies were killed was the only variable to increase the time needed for human extermination. The rate of transmission, the rate of the death of zombies, the rate of death by natural causes and the rate of infection resulted in radical effects upon the population classes, while the other variables either did not or only slightly affected the trends. The difference between the SZR and SIZR model was the resurrection rate of zombies. In the SZR model there was a slight effect, while in the SIZR model the variable had no effect. Regardless of the adapted variables, the zombies were able to completely eradicate the human population in a short period of time.

Discussion

The purpose of this study was to examine different types of models - basic, complex, realistic and imaginary - and determine the similarities and differences. The Lanchester models were derived from simple differential equations that were solved by hand, while the zombie models included complex differential equation which required the use of computer software. The zombie models were more complex than the Lanchester models because they contained multiple variables to depict relationships between 3 classes, while the Lanchester models only described a linear relationship using the fighting efficiency and number of forces as variables to describe the relationship between two population classes. Lastly, the zombie models depicted an imaginary situation based upon a disease model, while the Lanchester equations were realistic models used to depict war outcomes.

All the models examined in the study were discrete dynamical systems. Both types of models, Lanchester models and zombie models, required specific assumptions for the model to be accurate. The Lanchester model's assumptions were extremely specific and limited the type of battles they could be depict, which was why the original models have become outdated and mainly serve academic purposes. The zombie models, on the other hand, entailed general assumptions but required the zombie apocalypse to have occurred, which is impossible since it's a science fiction creation.

For the Lanchester models, I discovered that for the *X men* to win in the Aimed Fire model the fighting efficiency coefficient of the *X men* was greater than *n* squared times the fighting efficiency coefficient of the *Y men*. The solution related to the original Square Law that the Aimed Fire model was derived from, which had the number of forces squared, hence a greater impact upon the equation. Therefore, for the *X men* to win with a smaller number of forces, the *X men's* coefficient had to be greater than the original difference between the number of forces.

For the Area Aimed model, the number of forces and fighting efficiency coefficient had equal impact upon the time, winner, and losses of a battle. Hence, for the *X men* to win with a smaller number of forces, the *X men's* coefficient had to be greater than difference between the number of forces, $> n$.

For the zombie models, there were many similarities between the results of the sensitivity analysis for the SZR and SIZR, which is understandable since the differences in the differential equations was the introduction of an additional variable and population class. The transmission rate, the death rate of zombies, the death rate by natural causes and rate of infection greatly affected the trends of the population classes, while the birth rate and resurrection rate only produced slight changes if any changes at all. An increase in the death rate of zombies was the only variable to extend the time before total human eradication, but regardless of the adapted variables, the zombies were able to completely eradicate the human population in a short period of time.

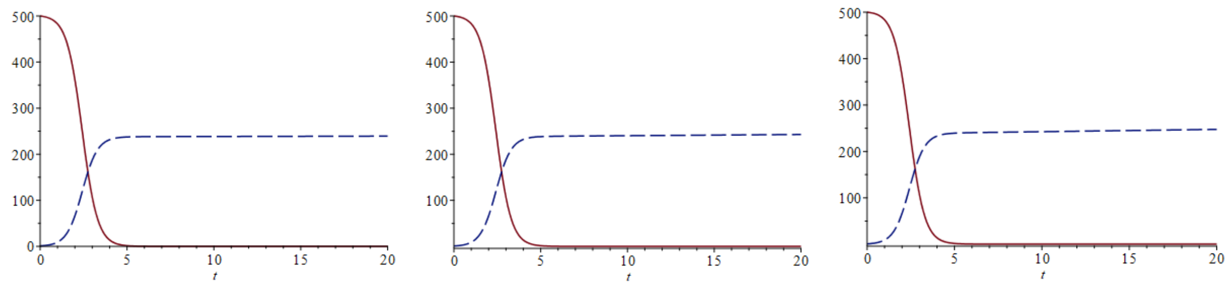
The study has provided me the opportunity to understand different types of mathematical models and the methods necessary to solve them. It also illustrated that mathematical models depicted a variety of situations, realistic as well as fictional. The models selected demonstrated the development of mathematical modeling from the simple linear equations used in 1915 in the Lanchester models to the complex differential equations used to describe the relationships between the different populations of the zombie apocalypse models in 2009.

If I were to continue my study, I would examine modern adaptations to the Lanchester Models and compare them to the original equations that they were derived from. I think it would also be beneficial to compare the SZR and SIZR models to models for infectious diseases and determine if there were a greater number of similarities. I would also like to include a variable to account for the difference in population density to determine if there is a significant difference between heavily populated and isolated areas. It would also be interesting to examine or create other science fiction-based models, such as a model to depict the werewolf or vampire virus.

Appendix

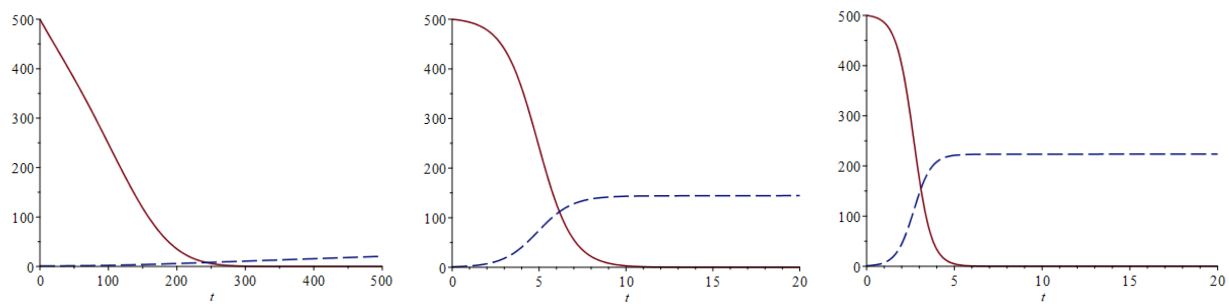
<u>Figure A</u> – Sensitivity Analysis of II using SZR model at t = 20 years

II = 0.1 II = 0.5 II = 1



<u>Figure B</u> - Sensitivity Analysis of β using SZR model

β = 0.005 at t = 500 β = 0.007 at t = 20 β = 0.009 at t = 20



<u>Figure C</u> – Sensitivity Analysis of δ using SZR model at t = 20 years

$\delta = 0.1$ $\delta = 0.5$ $\delta = 1$

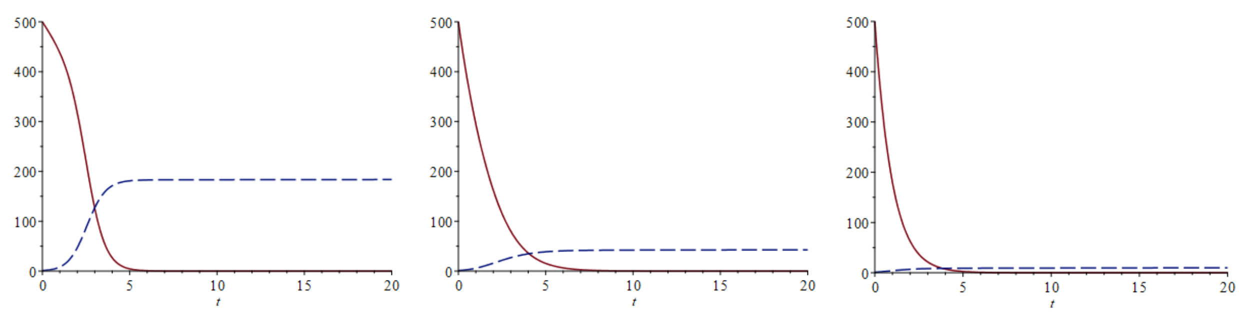


Figure D – Sensitivity Analysis of α using SZR model

α = 0.001 at t = 20 α = 0.004 at t = 20 α = 0.009 at t = 50

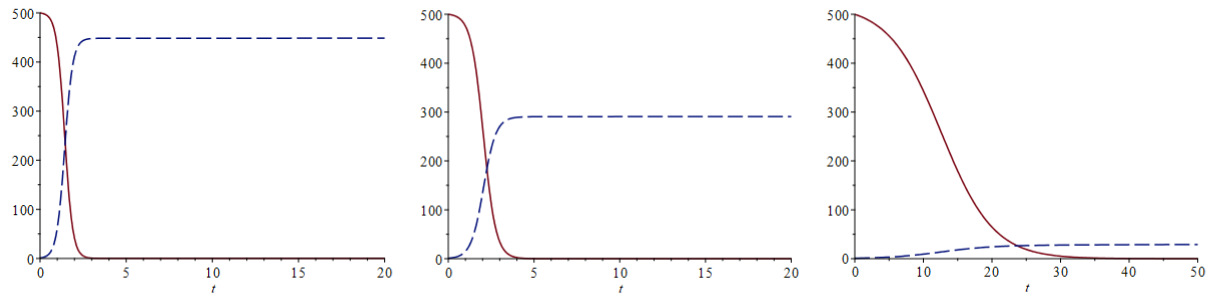


Figure E – Sensitivity Analysis of θ using SZR model at t = 10 years

θ = 0.1 θ = 0.5 θ = 1

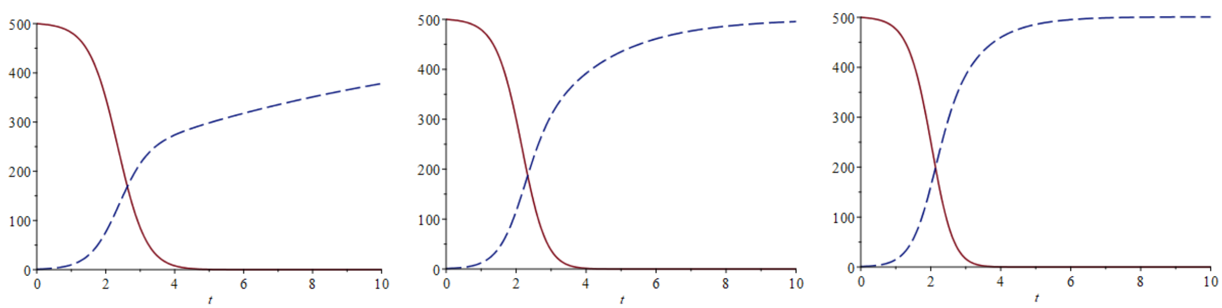
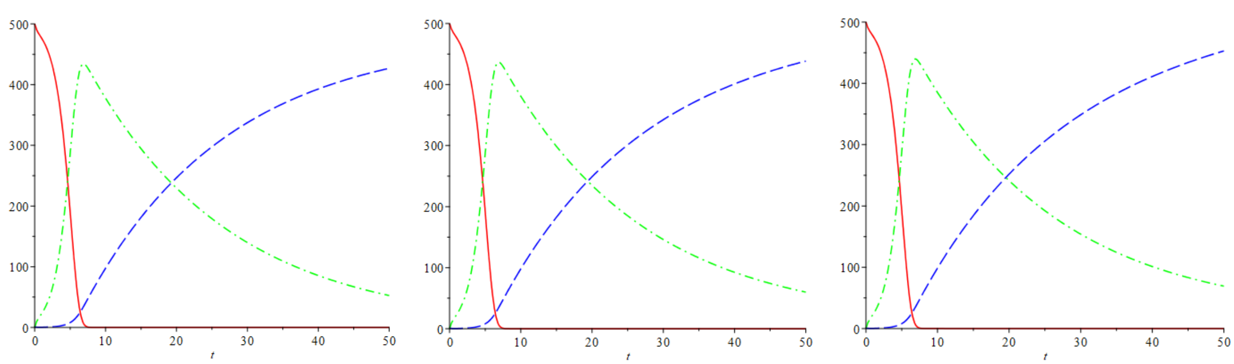


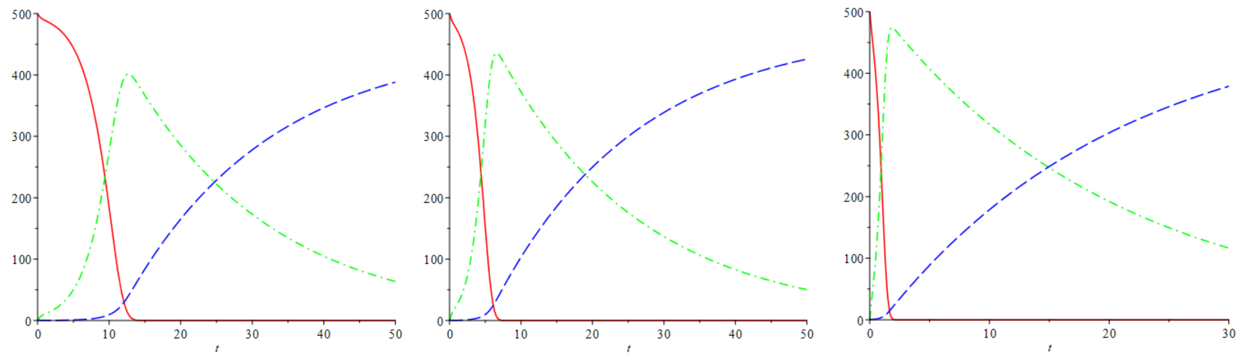
Figure F – Sensitivity Analysis of II using SIZR model at t = 50 years

II = 0.1 II = 0.5 II = 1



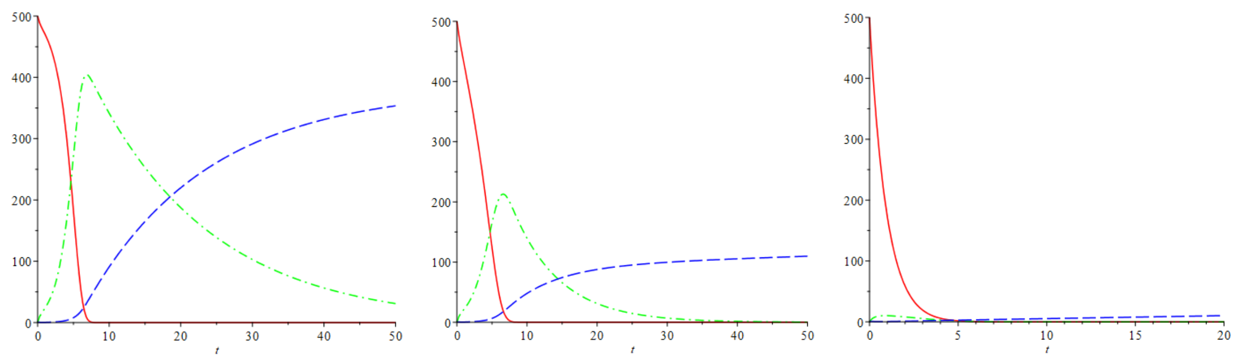
<u>Figure G</u> – Sensitivity Analysis of β using SIZR model

β = 0.05 at t = 50 β = 0.1 at t = 50 β = 0.5 at t = 30



<u>Figure H</u> – Sensitivity Analysis of δ using SIZR model

δ = 0.01 at t = 50 δ = 0.1 at t = 50 δ = 1 at t = 20



<u>Figure I</u> – Sensitivity Analysis of α using SIZR model

α = 0.01 at t = 100 α = 0.04 at t = 200 α = 0.08 at t = 500

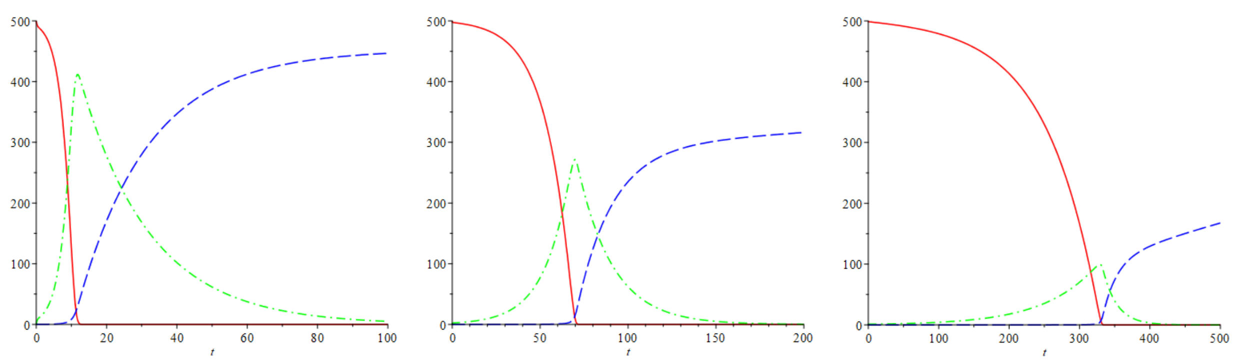


Figure J – Sensitivity Analysis of θ using SIZR model

θ = 0.1 θ = 0.5 θ = 1

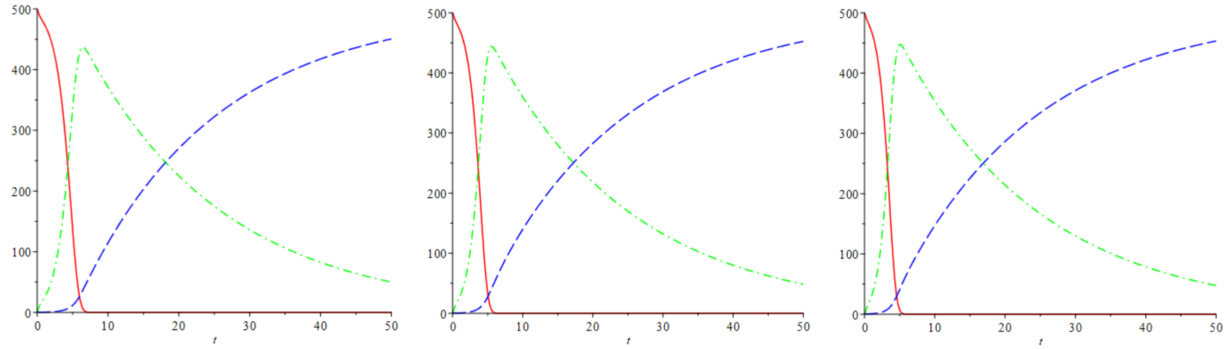
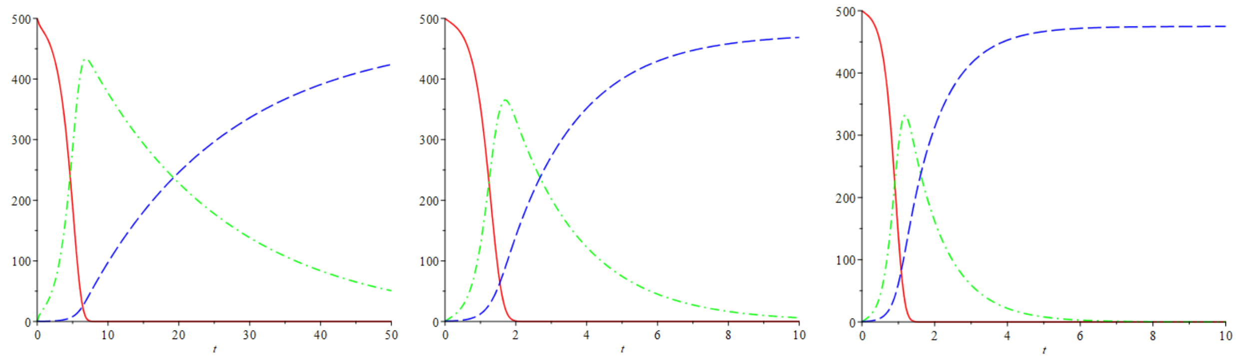


Figure K – Sensitivity Analysis of *p* using SIZR model

p = 0.05 at t = 50 *p* = 0.5 at t = 10 *p* = 1 at t = 10



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