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SET: The Probabilities and Possibilities

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Abstract

The card game SET involves finding groups of three cards called SETs. Choices are based upon the individual card characteristics, including shape, pattern, number, and color. Previously, the maximum number of cards that can be played without creating a SET has been determined as 20 cards by extensive computer work. This report further explored the probabilities and possibilities of the game. Using discrete mathematics and probability, we explored how many SETs are possible and what strategies led to the most points. Additionally, this project exercised undergraduate logic and reasoning to generalize the results in order to be applied in other fields of study. Furthermore, we investigated different methods of selecting and ordering cards, trying to find the minimum cards needed to guarantee a SET.

Introduction

Game theory is about strategy and prediction. It is used to predict outcomes of certain situations and to pick a strategy that can lead the player to his/her desired outcome. Game theory has a multitude of applications, including biology, business, economics, athletics, warfare, and politics [1]. Game theory can help analyze anything from whether a prisoner should plead guilty to predicting the chances of winning at a card game. Therefore, game theory is the logical analysis of situations of conflict and cooperation [1].

In Game Theory and Strategy by Straffin, a game is defined to be:

i.) *A situation in which there are at least two players.*

ii.) *Each player has a number of possible strategies, courses of action which he or she may choose to follow.*

iii.) *The strategies chosen by each player determine the outcome of the game.*

iv.) *Associated to each possible outcome of the game is a collection of numerical payoffs, one to each player. These payoffs represent the value of the outcome to the different players.* [1]

As will be shown in the "How to Play" section of this paper, SET qualifies under this definition as a game that can be analyzed by game theory. This project explores the strategy and tries to predict the possible outcomes of the game SET. The project will look at how players pick cards, what SETs players should be looking for, and how players should search for cards.

History of the Game

The game SET has origins in a German Shepherd Genome Project in Cambridge, England, 1974 [2]. Population geneticist Marsha Jean Falco was researching genes trying to discover if epilepsy in German Shepherds was inherited. Falco used cards to organize her data on the genes of the dogs, and used symbols to represent certain shared traits: different properties of the card would represent different gene combinations [2]. Falco would then search for certain SETs of cards to try to find the source of genetic epilepsy, if there was any. Finding the fun value in the cards, Falco taught her family how to play, and they convinced her to market the game [2].

Since its first marketing in 1990, SET has won over 25 Best Game Awards, including "Top 100 Games of 2005" from Games Quarterly Magazine and ASTRA (American Specialty Toy Retailing Association) Top Toy Pick in 1996 [3].

How to Play

SET can be played with one or more players. To set up the game, shuffle the deck of 81 cards and designate a dealer. The dealer will put out new cards as SETs are picked up. The dealer is also allowed to play. To begin, the dealer arranges 12 cards from the deck in a 3x4 rectangle, face-up on the table. Players must call SET before picking up the SET so that every other player may verify that it is a SET. Each SET is worth one point. If a player calls SET but does not actually have a SET, the player loses a point. The player with the most points when there are no more cards or SETs left is the winner of the game.

What is a SET?

Every card has a combination of color, pattern, symbol, and number. We call these broad categories the *characteristics* of the card. Therefore, we see every card has four characteristics. Within those characteristics are different options: color can be red, green, or purple; symbol can be ovals, squiggles, or diamonds; shading can be solid, open, or striped; and number can be one, two, or three symbols. We call these *properties*. Each characteristic has three properties. Since a SET is made up of three cards, to make a SET, each of the three cards in the SET have properties that are either all the same or all different.

Here are some examples of SET types:

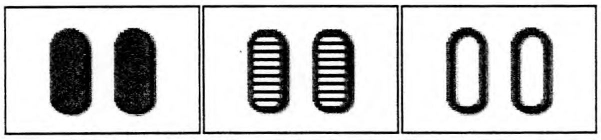

Figure 1: **Same** color, **same** symbol, **same** number, **different** pattern.

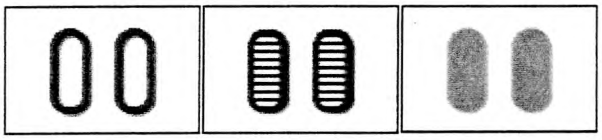

Figure 2: **Different** color, **same** symbol, **same** number, **different** pattern.

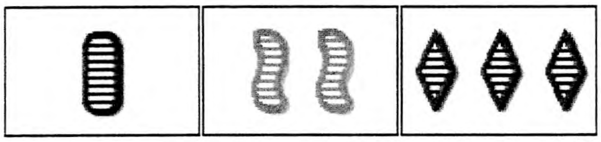

Figure 3: **Different** color, **different** symbol, **different** number, **same** pattern.

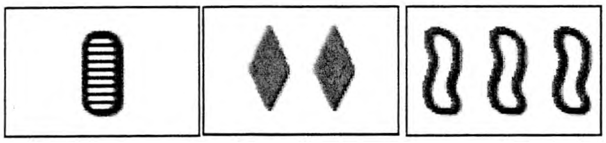

Figure 4: **Different** color, **different** symbol, **different** number, **different** pattern.

Any group of one to three same properties in any combination will be a SET as long as any characteristic that does not have the same properties has all different properties. Notice that there is not a SET for all four characteristics to be the same. That is because there is only one of every card, making each card unique in the deck.

Characteristic and Property Symbols:

The following symbols will be used within this project to refer to cards and SETs with certain properties of characteristics.

A card will be denoted by (Number, Color, Shape, Pattern)

a) For Number (**N**): 1,2, or 3
b) For Color (**C**): R for red, G for Green, or P for purple
c) For Shape (**S**): O for oval, D for diamond, or S for squiggle
d) For Pattern (**P**): s for solid, e for empty, p for striped.
e) For example, (1,G,D,e) is a card with one empty green diamond. This can also be written (1GDe), without commas.

When stating a SET's properties, (**N,C,S,P**)

a) + will stand for same properties

b) ~ will stand for different properties

c) For example, (**+N**,**~C**,**~S**,**+P**) is a SET with the same number of similar patterned different shapes of different colors.

Probability Background:

Before we begin, there are some definitions and theorems to be described. Among these are the basic principle of counting, permutations, and binomial coefficients.

The basic principle of counting is also known as the Fundamental Counting Principle (FCP). This principle states that if *r* experiments are to be performed are such that the first one may result in any of n_1 possible outcomes, and if for each of these n_1 possible outcomes there are n_2 possible outcomes of the second experiment, and if for each of the possible outcomes of the first two experiments there are n_3 possible outcomes of the third experiment, and if...then there is a total of $n_1 \cdot n_2 \cdots n_r$ possible outcomes of the *r* experiments. [4]

In simpler terms, the FCP tells us that we can take any number of different events, count the number of possible outcomes, and multiplying the number of possible outcomes from each event together to find the total amount of possible outcomes. We could easily prove this principle by drawing a tree diagram of our events and counting the sample space created from the data.

Another important word to understand is permutation. A permutation is an arrangement of objects. In other words, we are putting the objects in a certain order, then seeing how many other ways we can arrange the items so that we end up with a new order within our arrangement. We use *n factorial*, or *n!*, to show the permutations of an object. This proof uses reasoning from the basic counting principle. Suppose we have *n* objects. To pick the first object, we have n objects to choose from. To select the second object, we can no longer choose that first object, so we have (n-1) objects to choose from. Continuing this reasoning and the pattern

n(n-1)(n-2)...(3)(2)(1) emerges, or n!. [4]

The final definition to be discussed is the binomial coefficient $\binom{n}{r}$, called n choose r, represents the number of possible combinations of n objects taken r at a time. $\binom{n}{r} = \frac{n!}{(n-r)!r!}$.

At times, we need to look at the number of ways to choose a certain number of objects from a group. In general, n(n-1)···(n-r+1) represents the number of different ways of selecting r items when the order of selection is relevant. Each group of *r* items is actually counted more than once with this method. If fact, they are counted *r!* times. Therefore, we can modify our formula as $\frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{n!}{r!(n-r)!}$. [4]

Results

Theorem 1: There are 81 unique combinations of characteristics.

There are 3^4, or 81, unique combinations of the characteristics. See Table 1. The proof of this theorem is easy. Using the Fundamental Counting Principle, we can find the total sample space, also called the possible number of combinations where order does not matter.

Proof: We are given four categories: Number, color, shape, and pattern. Those categories each have 3 properties, therefore, by FCP, we have 3x3x3x3 total possible outcomes, also written as 3^4. ∎

1ROs	1GOe	1POp	2ROs	2GOe	2POp	3ROs	3GOe	3POp
1ROp	1GOs	1POe	2ROp	2GOs	2POe	3ROp	3GOs	3POe
1ROe	1GOp	1POs	2ROe	2GOp	2POs	3ROe	3GOp	3POs
1RDs	1GDe	1PDp	2RDs	2GDe	2PDp	3RDs	3GDe	3PDp
1RDp	1GDs	1PDe	2RDp	2GDs	2PDe	3RDp	3GDs	3PDe
1RDe	1GDp	1PDs	2RDe	2GDp	2PDs	3RDe	3GDp	3PDs
1RSs	1GSe	1PSp	2RSs	2GSe	2PSp	3RSs	3GSe	3PSp
1RSp	1GSs	1PSe	2RSp	2GSs	2PSe	3RSp	3GSs	3PSe
1RSe	1GSp	1PSs	2RSe	2GSp	2PSs	3RSe	3GSp	3PSs

Table 1: Every card in notation form. There are 81 cards.

Corollary: There are a^n *unique combinations of characteristics.*

Proof: For any combination where there are an equal amount of properties to each characteristic, there are a · a · a · a..... · a total number of possible outcomes, *a* being multiplied as many times as there are characteristics, where *a* is the number of different properties of each characteristic, and *n* is the number of characteristics. If there are *n* characteristics, then, there are a^n unique combinations. ▮

Theorem 2: Two arbitrarily drawn cards can only make a SET with one other card.

For an example, suppose you pick an arbitrary card, (2,P,O,p). Since each card is unique (and a card cannot make a SET with itself), you pick another arbitrary card, suppose (1,R,O,e). In order to make a SET, characteristics must be either all the same or all different. Already, we see characteristics (~**N**, ~**C**, +**S**, ~**P**). This means, in order to have a SET, the third card must follow the pattern laid down by the first two cards. Since same properties must be kept and there is only one property left to be different, we see we only have one choice to be the third choice, that is, (3,G,O,s). Since each card is unique, only one card in the deck will complete the SET.

Proof: We prove this by cases that look at the comparison of characteristics from the first card to the second, and the resultant third card.

Case 1: If the two cards chosen share no common characteristics and have four different characteristics, then the third card must also share no common characteristics and have four different characteristics. Since there are only three properties for each characteristic and two of the properties have been claimed by each the first card and the second card, for each characteristic, to complete the SET, the third card must contain the single leftover property of each characteristic. As there is only one choice for each property and, as each card is unique, there is only one possible card to complete the SET.

Case 2: If the two cards chosen share one common characteristic and three different characteristics, then the third card must also share one common characteristic and three different characteristics. The common characteristic will be the same as the first and second card. For the different characteristics, since there are only three properties, for each characteristic and two of the properties have been claimed by each the first card and the second card for each characteristic, we will have to complete the SET with the single leftover property of each of the three different characteristics. We have shown there is only one choice for each property and as each card is unique, there is only one possible card to complete the SET.

Case 3: If the two cards chosen share two common characteristics and two different characteristics, then the third card must also share two common characteristics and two different characteristics. Using the same reasoning as the first two cases, the common characteristics will be the same as the first and second card. For the different characteristics, since there are only three properties for each characteristic and two of the properties have been claimed by each the first card and the second card for each characteristic, we will have to complete the SET with the single leftover property of each of the two different characteristics. We have shown there is only one choice for each property and as each card is unique, there is only one possible card to complete the SET.

Case 4: If the two cards chosen share three common characteristics and one different characteristic, then the third card must also share three common characteristics and one different characteristic. Once again we see for the three common characteristics, the property will be the same as the first and second card. For the different characteristics, since there are only three properties for each characteristic and two of the properties have been claimed by each the first card and the second card for each characteristic, we will have to complete the SET with the single leftover property for the different characteristic. We have shown there is only one choice for each property and as each card is unique, there is only one possible card to complete the SET.

Therefore, two arbitrarily drawn cards only make a SET with one other card. ∎

Here is another approach by David Van Brick that proves through the characteristics as well, that is less general, but easier to apply to the game of SET. The proof is by construction, in four steps.

1. *If the two cards are of the same shape, then the third card has that shape; otherwise, it must have whatever shape is not on either of the two first cards.*
2. *If the two cards are of the same shading, then the third card has that shading; otherwise, it must have whatever shading is not on either of the two first cards.*
3. *If the two cards are of the same color, then the third card has that color; otherwise, it must have whatever color is not on either of the two first cards.*
4. *If the two cards are of the same number, then the third card has that number; otherwise, it must have whatever number is not on either of the two first cards.*

   If you go through these four steps, you will find the unique third card that completes any two cards' SET. ∎ [5]

Corollary: If we have n cards in a SET and have chosen n-1 cards for our SET, there is only one possible solution for our final card.

Proof: Suppose there are *n* number of properties for *c* amount of characteristics. Then there are *n* number of cards in a SET (Notice that the number in a SET will always be the same as the number of different properties of a certain characteristic in order to be able to use every property in a "different" property SET). If we have drawn all the cards except for one, we have chosen n-1 properties from n properties in a SET. This means we have chosen n-(n-1) properties, which is n-n+1, or just 1 property in each characteristic left. And we see no matter the amount of characteristics, if we have all but one of the cards, there is only one property left for each characteristic, creating a singular result for our final card in the SET. ▩

Theorem 3: An arbitrary card has 40 possible SETs.

Originally, it seems as though this proof will be as simple as $\frac{\binom{81}{3}}{3!}$. We divide by 3!, since every SET would be counted 6 times due to permutations: i.e. (1ROs), (1ROp), and (1ROe); (1ROs), (1ROe), and (1ROp); (1ROp), (1ROs), and (1ROe); (1ROe), (1ROp), and (1ROs); (1ROp), (1ROe), and (1ROs); and (1ROe), (1ROs), and (1ROp) are all the same set by (+**N**,+**C**,+**S**,~**P**) . However, as we see later in proposition 1, we cannot just pick three cards out at random, as this method suggests, and expect them to make a SET. Therefore, we look for other methods.

Proof: Arbitrarily draw a card. From the deck of 81, there are now 80 cards. Pick another arbitrary card. From theorem 2, there is only 1 card option to complete the SET. Those two cards will not be able to be used in any other SET, so we remove them from the deck. Now there are 78 cards. Continuing to make SETs will use 2 cards at a time until they are all used. Therefore, an arbitrary card can make 80/2 or 40 possible SETs. See Table 2 below.

Selected:	1ROs						
1GOe	1POp	2POe	3GOp	1RSp	1RDe	2GSp	3PDe
1POe	1GOp	2POs	3GOs	1RDp	1RSe	2RDe	3RSp
1PDe	1GSp	2PDe	3GSp	1GDp	1PSe	2RSp	3RDe
2PDp	3GSe	2PDs	3GSs	1PSp	1GDe	3RDp	2RSe
3PDp	2GSe	2PSp	3GDe	1GSs	1PDs	3ROp	3ROe
2GDe	3PSp	2PSe	3GDp	2ROp	2Roe	1ROp	1ROe
1GOs	1POs	2PSs	3GDs	2GOe	3Pop	1RDs	1RSs
2ROs	3ROs	2GDs	3PSs	2POp	3GOe	1PDp	1GSe
1GDs	1PSs	2GDp	3PSe	2GOs	3Pos	2RDs	3RSs
2RDp	3RSe	2GSs	3PDs	2GOp	3Poe	2RSs	3RDs
						Total:	40 SETs

Table 2: One card is selected, showing 40 SETs made using all of the other cards.

How do we know this works all the time? It seems as though at least one of those cards would have been used in another SET beforehand, keeping every card from being used in perfect SETs. Luckily, we know we can make exactly 40 SETs from any one card from Theorem 2. Since we have used the first card already, any card we choose second will only have 1 third choice. Suppose we have used that third card already. By theorem 2, the only card we could have used it for would be the second card. Therefore, we see there are no overlaps within those 40 SETs made from one card. The same is not true when we release the SETs from being made with one card and look at all the SETs at the same time, which is mentioned later in the paper.

Corollary: An arbitrary card has $\frac{n-1}{c-1}$ *possible SETs.*

Proof: Let n be the number of cards in a deck. Then the number of cards available to make SETs with after choosing one is n-1. Now let c be the number of cards in a SET. Because we have already chosen one card of the set we now have c-1 cards to find in the SET. So we can put 1 card with n-1 different cards into c-1 sized groups. This will be counting each group, c-1 extra times, once for every card from the first to the "c-1"st card. To compensate for this, we arrive at our final formula:

$$\frac{n-1}{c-1}$$

Theorem 4: You are most likely to make a SET with 1 same property characteristic, and 3 characteristics with all different properties.

Recall from section "What is a SET?" that there are four different ways to make a set based upon the properties of each characteristic of the cards, such as color or shape, and those properties such as having 1,2, or 3 ovals, diamonds, or squiggles. What we want to do is to look at the breakdown of SETs to see which SET-type occurs the most. Our hypothesis would be all different would have the most because there seems to be more variety in the SETs. However, our hypothesis was proven false. The results showed that permutations were the key to the most SETs.

Proof: To start this proof, we will use the FCP to find the number of cards that will make up the SETs of each type of SET, assuming we have already drawn the first card.

> **1 Same, 3 Different:** The first card has already decided the property of our first characteristic, so we have 1 choice for that and that is our same characteristic. For our

second characteristic, we have two choices, because we cannot choose the same property as the first card in order to make a different characteristic. The same reasoning holds for the third and fourth characteristics, and we end up with $1 \cdot 2 \cdot 2 \cdot 2 = 8$ card options.

2 Same, 2 Different: $1 \cdot 1 \cdot 2 \cdot 2 = 4$ card options.

3 Same, 1 Different: $1 \cdot 1 \cdot 1 \cdot 2 = 2$ card options.

4 Different: $2 \cdot 2 \cdot 2 \cdot 2 = 16$ card options.

At this point, it looks like 4 different characteristics will have the most card options, but now we need to look at permutations. To find the permutations, we need to look at a binomial coefficient$\binom{n}{k}$, where n will be the number of positions and k will be the number of same properties.

1 Same, 3 Different: $\binom{4}{1} = \frac{4!}{1!(3!)} = 4$ permutations

Now, we take the card options from earlier and multiply by the number of permutations.

$8 \cdot 4 = 32$ card options or 16 possible SETs.

2 Same, 2 Different: $\binom{4}{2} = \frac{4!}{2!(2!)} = 6$ permutations

$4 \cdot 6 = 24$ card options, 12 SET options.

3 Same, 1 Different: $\binom{4}{3} = \frac{4!}{3!(1!)} = 4$ permutations

$2 \cdot 4 = 8$ card options, 4 SET options.

4 Different: $\binom{4}{4} = \frac{4!}{4!(0!)} = 1$ permutation

$16 \cdot 1 = 16$ card options, 8 SET options.

As a check, we made sure these SET options all add up to 40 (from theorem 3), and they do.

See Table 3 below to see how this looks.

Selected: 1ROs

3 Same	1ROp	1ROe
	2ROs	3ROs
	1GOs	1POs
	1RDs	1RSs
	Total:	**4 SETs**

1 Same	2RDp	3RSe
	2GOp	3POe
	1PDe	1GSp
	2POe	3GOp
	1PDp	1GSe
	2POp	3GOe
	2PDs	3GSs
	2GOe	3POp
	2GDs	3PSs
	2GSs	3PDs
	2RDe	3RSp
	2RSp	3RDe
	2RSe	3RDp
	2PSs	3GDs
	1GDp	1PSe
	1PSp	1GDe
	Total:	**16 SETs**

2 Sames	2GOs	3POs
	1POe	1GOp
	1GOe	1POp
	2POs	3GOs
	1GSs	1PDs
	1RSp	1RDe
	2RSs	3RDs
	2RDs	3RSs
	1GDs	1PSs
	2ROe	3ROp
	1RDp	1RSe
	2ROp	3ROe
	Total:	**12 SETs**

No Sames	2GDe	3PSp
	2PDp	3GSe
	2GSe	3PDp
	2PDe	3GSp
	2GDp	3PSe
	2GSp	3PDe
	2PSp	3GDe
	2PSe	3GDp
	Total:	**8 SETs**

Table 3: The SETs made from one card from Table 2 grouped by type.

Corollary 1: 81-Card Deck SET breakdown.

In the whole deck, we will have *t*, the number of SETs in a particular type, times 81, the number of cards in the deck, divided by 3 to account for repeats. That is, $\frac{t(81)}{3}$.

Therefore:

1 Same, 3 Different $\frac{16(81)}{3}$**:** = 432

2 Same, 2 Different: $\frac{12(81)}{3}$ = 324

3 Same, 1 Different: $\frac{4(81)}{3}$ = 108

4 Different: $\frac{8(81)}{3}$ = 216

We check ourselves again by seeing if they add up to 1080 SETS (theorem 5), and they do. ∎

Corollary 2: There are $\frac{d(\mathrm{m}-1)\mathrm{n}!}{2(k)!(n-k)!}$ *SETs of k-same, d-different SET-type for a particular card.*

Proof: Suppose we have already arbitrarily drawn a card. Let *d* be the number of different characteristics for the intended SET-type (i.e. the three-sames-and-one-different SET listed above would give *d* a value of 1). Let *k* be the number of same characteristics (i.e. the three-sames-and-1-different SET listed above would give *k* a value of 3). Let *m* be the number of properties within each characteristic (i.e. for characteristic color, there are 3 properties: red, purple, and green) and m should be a constant for all characteristics.

We know from our theorem 4 that this formula will be the FCP of the number of cards that will make up the SETs of each type of SET (That is, when we are trying to make a SET, we have four different types of SETs we can make with that card (See "What is a SET?")) times the permutations.

For the FCP, for every same-type characteristic, we must continue to use that first chosen property, which means we have 1^k options. For every different-type characteristic, we must choose another different property of that characteristic for each the other two cards that will complete our SET. We cannot use our first property option, therefore our different characteristic options will be given by d(m-1).

We arrive at 1^k d (m-1).

Now we look at the permutations. Let *n* be the number of positions in the permutation and *k* again be the number of same properties. We know our permutations will be given by $\binom{n}{k}$.

1^k d (m-1) $\cdot \binom{n}{k}$

$$= 1^{\mathrm{k}}\, d\,(m-1)\,\cdot \frac{n!}{k!\,(n-k)!}$$

$$= \frac{(\mathrm{d}\,(\mathrm{m}-1))\mathrm{n}!}{(k)!(n-k)!}$$

We have now arrived at the number of cards that will make up each SET type. The final step will be to group those cards by 2 in order to pair with our originally chosen card in SETs, rather than by number of card choices.

Therefore, $\frac{(d\,(m-1))n!}{2(k)!(n-k)!}$. ∎

Theorem 5: There are 1080 possible SETs.

Proof: For this calculation, consider Theorem 3. If any card has 40 possible SETs, then the deck of 81 cards has 81x40, or 3240 SETs. However, consider that there will be repeats within that. This calculation has counted every SET three times, once for each of the three cards in the SET. Therefore, the real number of possible SETs is 3240/3, or 1080 SETs. ∎

Corollary: There are $\frac{n^2-n}{c^2-c}$ *possible SETs in the entire deck.*

Proof: Consider Theorem 3 Corollary 1. If any card has $\frac{n-1}{c-1}$ possible SETs, where n is the number of cards in the deck and c is the number of cards in a SET, then the entire deck has a total of $n(\frac{n-1}{c-1})$ SETs, or $\frac{n^2-n}{c-1}$ SETs. However, this formula counts every set c times, so we need to divide our formula by c, finding our final formula as:

$\frac{n^2-n}{c^2-c}$ ∎

Proposition 1: You cannot arbitrarily pick sets and use all the cards.

During a normal game, players pick the first SETs they see. This can be described as picking arbitrary SETs. And the end of the game, there are often cards left over in which no SETs are present. Thus, you cannot arbitrarily pick sets and use all the cards.

How many cards can be laid down without a SET being present?

It is best to start with a definition of what is not a SET. Three cards are not a SET anytime "two cards are and one is not." For example, if two cards have same or different properties for at least one characteristic, and the third card does not following the established pattern (whether it be different or same), then the cards do not make a set.

One way of guaranteeing a SET will not be present is to only pick 2 of the 3 types of properties from each characteristic (2^4=16). Therefore, there can never be three cards with all the same or all different characteristics. Could there only be 16 cards without SETs? Counter example: Figure 5. This method does not account for every card that could be picked that would guarantee a SET is not present.

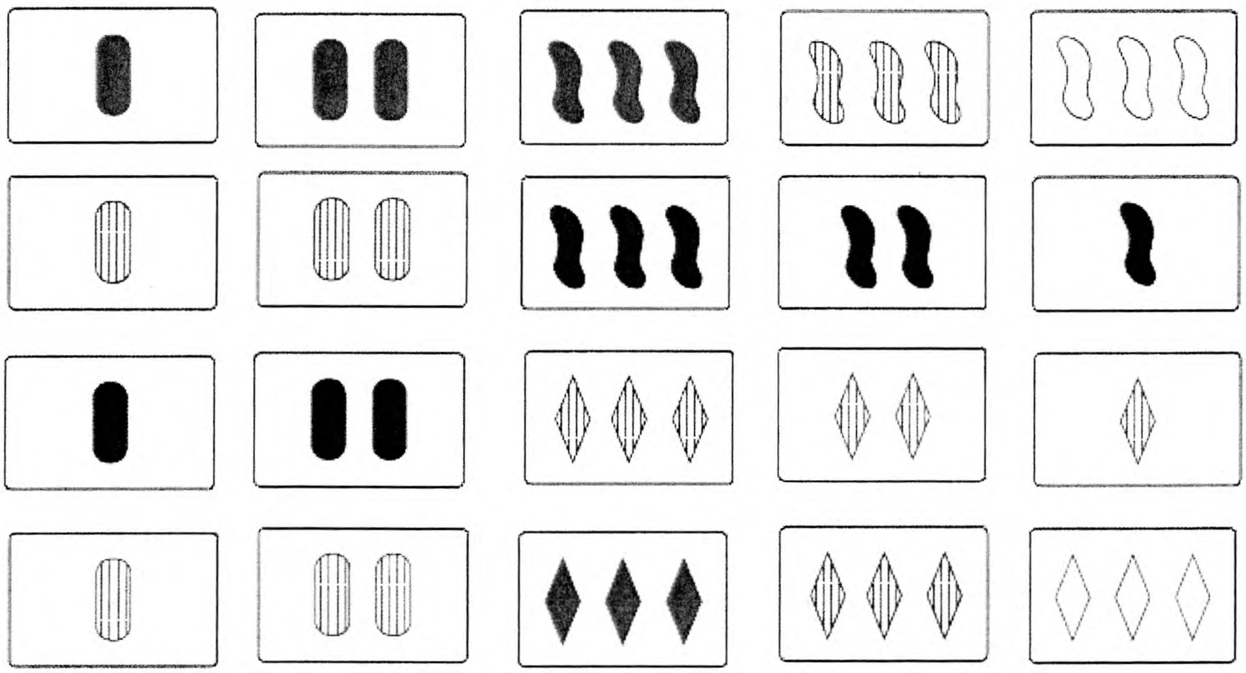

Figure 5: A selection of 20 cards with no SETs [6]

The SET Company offers the table in Figure 6 to show how many cards can be placed without a SET being present:

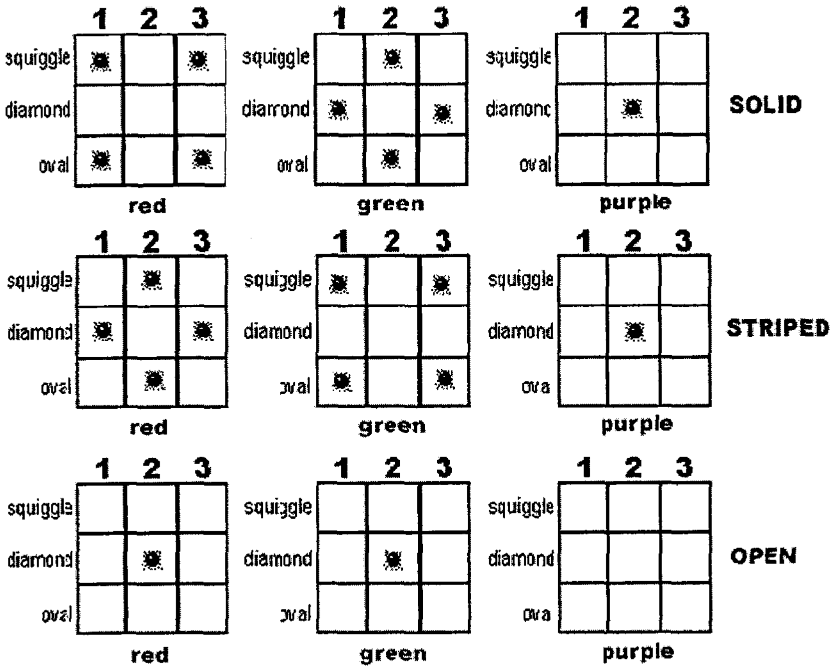

Figure 6 A mapping of 20 cards with no SETs. [7]

In Figure 6, a SET is created whenever there are three dots in a row in a single box (i.e. horizontally, vertically, or diagonally) or whenever a dot is repeated in the same section of the 3-by-3 boxes (i.e. the middle of any three will be a SET or 3 separate boxes making a horizontal, vertical, or diagonal line of dots). The question remains as to the number of cards that guarantee a SET is present.

How many cards guarantee a SET?

Guaranteeing a SET will be defined as the lowest number of cards where there is at least one SET present and will always be present.

For this exploration, we will refer to the tree diagram in figure 7 below.

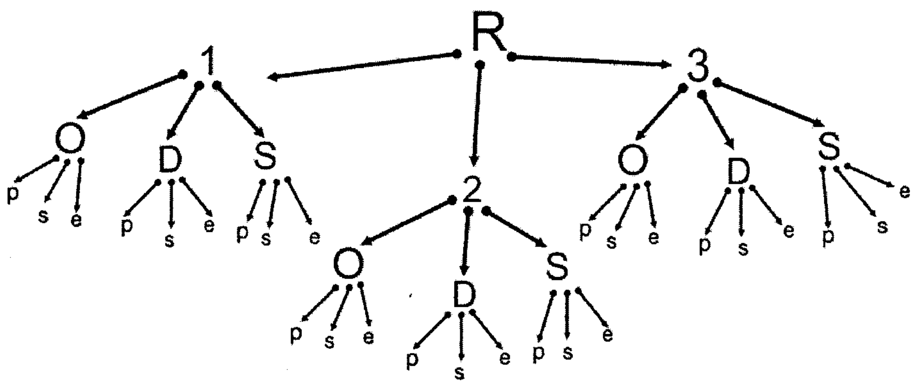

Figure 7: A tree diagram showing all red cards.

The tree diagram in figure 7 shows 27 cards, all the cards in one color. In the whole deck, the tree diagram would consist of three such trees, one more for each of the colors green and purple.

Like in any tree diagram, our last section gives us the sample space. It is where we can look to see every possible option. Now, we want to focus on one particular branch, shown below:

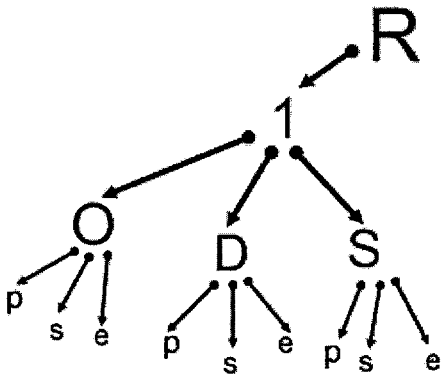

Figure 8: A tree diagram showing all red cards of only a single item.

Figure 8 shows our card breakdown when two characteristics are restricted. From this, we can derive the following theorem.

Theorem 6: *When two characteristics are restricted, 5 cards guarantee a SET.*

Proof: Let any two characteristics be restricted. Therefore, there are two unrestricted characteristics, three properties each. By theorem 1 corollary, there are now 3^2 cards, or nine cards. Choose any two cards. These cards automatically have two same characteristics, and any SET made with these cards will have to be made with the remaining 7 cards (they are the only cards that will have the same first characteristic and the same second characteristic). From theorem 2, these two cards will only make a SET with one other card. We shall set that card to the side, in order to find the maximum number of cards it will take to guarantee a SET. There are now 2 cards chosen and 6 remaining cards to choose from. We choose another card. Now, we have $\binom{3}{2}$ possible pairs, or 3 possible pairs, one we already chose and two new ones. Therefore, we must remove two cards that will make SETs and put them aside as well. We now have 3 chosen cards and 3 remaining cards to choose from.

We continue the pattern:

$\binom{4}{2} = 6$, subtract the pairs we have already made, 3, and we find that three cards will make a SET with our chosen cards. Now we have chosen 4 cards and have set aside a total of 6 cards (more than the nine we started with suggesting overlap within the SETs. This will not affect the outcome because the overlap is in the cards we are not choosing).

Let us chose another card. Wait! The only cards that we can choose now are the ones we set aside because they complete SETs. At this point, any fifth card chosen will guarantee a SET. ∎

Theorem 7: When one characteristic is restricted, 13 cards will guarantee a SET.

Proof: Consider theorem 6: 5 cards guarantee a SET when two characteristics are restricted. If we release one of those restricted characteristics, our sample space becomes 3^3 cards big, or 27 cards. We now have three identical branches in our tree to work with. In order to guarantee a SET, we must guarantee that at least 5 cards have gotten down at least one of those branches. In better terms, we have at least 5 cards that have the same second characteristic. We have three characteristics. If we put the maximum cards without guaranteeing a SET in each of them, we have $3 \cdot 4$, or 12 cards without guaranteeing a SET. We easily see, then, that any one card added will mean we will have at least 5 cards with at least two same characteristics, which from theorem 6, guarantees a SET. Therefore, 13 cards will guarantee a SET when only one characteristic is restricted. ∎

Theorem 8: When no characteristics are restricted, 37 cards will guarantee a SET.

Proof: Consider theorem 6 and the logic that took us from two restricted to one restricted characteristic. We will consider the same reasoning to prove the case of no restrictions. We need to show how many cards it will take to guarantee 13 cards, all with the same first characteristic. From theorem 1, we know we now have 81 cards and 3 properties. We can easily see that 36 cards can only guarantee as many as 12 cards in each characteristic with no SETs present. However, at 37 cards, we can guarantee that at least one of the first characteristics has the 13 cards needed to guarantee a SET. Therefore, we see 37 cards guarantees a SET. ∎

The problem with theorems 6 and 7 is that they fail to take into consideration different-different-different-same SET-types and different-different-different-different SET-types.

Unfortunately, we do not believe that anything short of extreme calculations done by a computer will give us a definite proof in these two cases.

Experimenting with the cards found that 10 cards became our guaranteed-SET-amount when one characteristic was restricted. From this find, we know that at least 28 cards would guarantee a SET if we unrestricted all characteristics and we can speculate that seven cards made SETs across the colors (the different-different-different-different SET-type), since we know 21 is the actual number of cards to guarantee a SET.

Conclusions

We see that just a small portion of characteristics and properties can make choosing three specifically related cards incredibly complex. With 1080 options to make SETs, there are a lot of possibilities when playing. This gives us lots of options for how a game will go, too many for a person to try. This leads me to believe that if we want any more significant progress from this point, we will need either much more complex mathematical abilities or computer-help, especially in the case of the number of cards that will guarantee a SET.

The most surprising result turned out to be the breakdown of SET-type. This will help any player trying to find SETs quickest. Outside of this game or other games related to it, we do not see any further-reaching consequences. The most interesting result was that 2-cards could only make a SET with one other card. While a simple concept once known, before this discovery, the game is at least 79-times more difficult to play and to try to prove.

This project has uncovered the math behind simple genetic tracking. Just like we found when trying to guarantee a SET, when working with the genes of a species, the complexity makes us turn to computers to do the intricate calculations. As such, this project can be explored more by looking at contemporary methods that have been developed alongside genetic research.

This project has discovered more about how SETs are made, where there will be SETs, and when there will be SETs. We have explored many probabilities of types of sets. This project also visited the main proof of how many cards guarantee a SET. We were hoping to be able to logic-out this proof based upon our earlier finds, but there are too many unknowns to make a definitive proof just based on what we found. There may still be methods of picking cards to find the minimum amount of cards that need to be played in order to guarantee a SET. We would like also to look at the graduate-level math of that MacLagen and Davis report [6] and try to understand their methods.

We would have liked to explore how our results could have been applied to game theory and genetic theory, unfortunately there was insufficient time to extend this investigation. This project also has room to expand toward interdisciplinary applications. Our formulas may be able to be used in other fields and in other similar games. Perhaps the formulas could help advertisers better select the ads to place based upon previous items bought or viewed. However, we were always more interested in just really knowing how this particular game, SET, works and the math behind our entertainment.

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