Spring 4-12-2022

Calculation of the Fundamental Electric Charge with a Millikan Oil Drop Apparatus

Noah Fuge D'Antonio
New Mexico Institute of Mining and Technology, Noah.D'Antonio@student.nmt.edu

Calculation of the Fundamental Electric Charge with a Millikan Oil Drop Apparatus

Anonymous

April 12^{th}, 2022

Abstract

Using a Millikan oil drop apparatus, information obtained from drops falling at terminal velocity and suspended between capacitor plates can be used to calculate the charge q on a oil drops. Using a large number of drops, we attempt to determine the value of the fundamental electric charge e that best fits the equation $\frac{q}{e} = n$, where n is an integer. By attempting to minimize the square of the non-integer residual for all drops, the best fit value for e obtained was $e_{measured} = (3.8 \pm 0.4 * 10^{-19})\ C$. As the tabulated and defined value of $e_{tabulated} \approx 1.602 * 10^{-19} C$ is far outside the bounds of random error, there was likely significant systematic error with the experimental setup or issues with data analysis. Potential sources of experimental error contributing to this discrepancy are examined, and the drawbacks of using a simple residual minimization formula are discussed.

Introduction

The fundamental electric charge e is the smallest unit by which all electrical charge is measured. Both the proton and the electron have charge e, and aside from a few special cases such as the fractional Hall effect (Laughlin, 1983), *all* electrical charges are made of an integer number of fundamental electric charges (Foot, Lew, & Volkas, 1993), whereby $q = ne$, $n \epsilon \mathbb{Z}$.

One of the most well known ways to measure the fundamental electric charge is the oil drop method. The first largely successful use of this method to determine e was first completed by Robert A. Millikan in 1911, and was an improvement on previous attempts to find the electric charge using charged particles of water or alcohol (Millikan, 1911). By measuring the velocity of a drop of oil under an electric field, Millikan could determine the charge of the drop. Using sources of ionizing radiation, he could view changes in the charge when it either gained or lost an ion, and the affects that had on electromagnetic forces acting on the drop. After correcting for several sources of error, particularly deviations from Stokes law for small particles, he was able to calculate an electric charge value of $e \approx 4.891 * 10^{-10}\ ESU \approx 1.631 * 10^{-19}\ C$. Further experiments by

Millikan (Millikan, 1913) and others were able to further increase the accuracy of this measurement, and in 2018 the fundamental charge was *defined* to be $e \equiv 1.602176634 * 10^{-19}$ (Tiesinga et al., 2018)

Without the ability to use radiation to create an ionizing environment so that oil drops easily loose and gain electrical charge over time, this experiment takes a slightly different approach. Recognizing that the motion of the drop is dependant on the sum of multiple forces including gravity, drag, the buoyant force, and the electromagnetic force, it is possible to set up the apparatus to determine the radius, mass, and charge of the drop (Hogan & Hasbun, 2016).

Methods

A United Scientific Supplies Millikan Oil Drop Apparatus 01 was used to determine the fall time and voltage required to suspend an oil drop in order to determine the radius, terminal velocity, and charge on each of the 36 drops. The voltage readout was calibrated by comparing the values of a voltmeter and comparing to voltages from 0 to 315 volts on the apparatus readout. Using a linear least squares fit (Taylor, 1997), the integer readout values were corrected using the formula $V_{voltmeter} = 0.9924V_{readout} + 0.2772$. The spacing of the screen divisions were calibrated by measuring the screen spacing, as well as a wire of known diameter $d_{wire} = 0.266 \pm 0.002\ mm$ (see Appendix for details).

The values of the constants used in this lab are given in **Table I** below. Note that the location of **Table II** is in the Appendix at the end, and that any temperature dependant constants are calculated at 294.3 K:

Quantity	Symbol	Value	Error	Source
Viscosity	η	$1.823 * 10^{-5}\ N\ s\ m^{-2}$	$6.9 * 10^{-9}$	Allen & Raabe
Gravity Acceleration	g	$9.79\ m\ s^{-1}$	N/A	Pavlis et al.
Density of Oil	ρ	$981\ kg\ m^{-3}$	± 5	Manufacturer
Density of Air	ρ_{air}	$1.025\ kg\ m^{-3}$	$4.6 * 10^{-3}$	Calculated
Pressure	P	$88600\ Pa$	± 100	Measured
Stokes Correction	b	$8.2 * 10^{-3}\ Pa\ m$	N/A	Hogan & Hasbun
Stokes Correction	C	$1 + \frac{b}{rP}$	N/A	Calculated
Capacitor Separation	d	$5 * 10^{-3}\ m$	± 0.01	Manufacturer
Radius of Drop	r	Table II	Table II	Calculated
Mass of Drop	m	Table II	Table II	Calculated
Balancing Voltage	V	Table II	Table II	Measured
Charge on Drop	q	Table II	Table II	Calculated

Table I: Table of different constants used in the paper's calculations. The error of g and b was not known well enough to tabulate.

Using a sign convention where forces downwards are negative, a drop of mass m has a gravitational force $F_g = -gm$ acting on it, and is opposed by the drag force and buoyant force. The drag force is given by Stokes law applied to a spherical surface under laminar flow, $F_d = 6\pi r\eta v$. The buoyant force is $F_b = gm_{air}$, where m_{air} is the mass of displaced air. A reasonable assumption to make is that the density each drop is independent of their radius (Millikan, 1913). From this, the mass of the drop is a product of the volume and density $m = \frac{4}{3}\pi r^3 \rho$, and the mass of air displaced $m_{air} = \frac{4}{3}\pi r^3 \rho_{air}$
At terminal velocity, there is no net acceleration and the forces thus sum to zero:

$$F_g + F_d + F_b = 0 \rightarrow 6\pi r\eta v - \frac{4}{3}\pi r^3 g(\rho - \rho_{air}) = 0$$

The largest complication to this formula is the assumption that Stokes law applies to droplets of all sizes. It is well known that droplets of smaller size violate this assumption (Allen & Raabe, 1985), and therefore a correction to the viscosity of air is required. This correction factor has many forms, and the one chosen for this paper is given as $\eta_{corrected} = \frac{\eta}{C}$, where $C = 1 + \frac{b}{rP}$(Hogan & Hasbun, 2016).
The equation then becomes:

$$6\pi r\frac{\eta}{C}v - \frac{4}{3}\pi r^3 g(\rho - \rho_{air}) = 0 \rightarrow \frac{r^3}{r}C = \frac{18\eta v}{4g(\rho - \rho_{air})}$$

$$r^2 + r\frac{b}{P} - \frac{9\eta v}{2g(\rho - \rho_{air})} = 0$$

This may be trivially solved with the quadratic equation:

$$r = \frac{1}{2}\left(-\frac{b}{P} \pm \sqrt{\left(\frac{b}{P}\right)^2 - 4\frac{9\eta v}{2g(\rho - \rho_{air})}}\right)$$

Taking the square root to be positive for physicality and simplifying:

$$r = \sqrt{\left(\frac{b}{2P}\right)^2 - \frac{9}{2}\frac{\eta v}{g(\rho - \rho_{air})}} - \frac{b}{2P}$$

The mass of the oil drop may then be calculated: $m = \frac{4}{3}\pi r^3 \rho$
Given the electric force between capacitor plates is (Griffiths, 1999) $F_e = qE = q\frac{V}{d}$, the charge q was calculated in this lab from the voltage at which $F_e + F_g = 0$. From this, $q\frac{V}{d} - mg = 0 \rightarrow q = \frac{mgd}{V}$.

Results

The values of mass, radius, terminal velocity, charge, etc. for each drop are given in **Table II** found in the Appendix at the end of this paper.
It is not immediately clear how the calculated charge relates to the number of fundamental electric charges on each drop of oil. However, given the assumption that the the oil drops contain an integer number of fundamental electric

charges, we know $n = \frac{q}{e}$, where n is an integer. One strategy to determine the fundamental electric charge is to find which value of e most closely divides q into integer amounts. We may assign a single number to the residual of these numbers by summing over the square of the residual of each drop:

$$S^2 = \sum_{i=1}^{36} (\frac{q_i}{e} - round(\frac{q_i}{e}))^2$$

In order to place reasonable bounds on this equation, we will assume e is no smaller than $\frac{1}{100} q_{min}$, and no larger than q_{max}, where q_{min} and q_{max} are the drops with the minimum and maximum charges respectively.

As seen in **Figure I** below, the value where this is minimized is found at $e = 3.77 * 10^{-19}$ C, and gives a sum of squares residual value of 0.376.

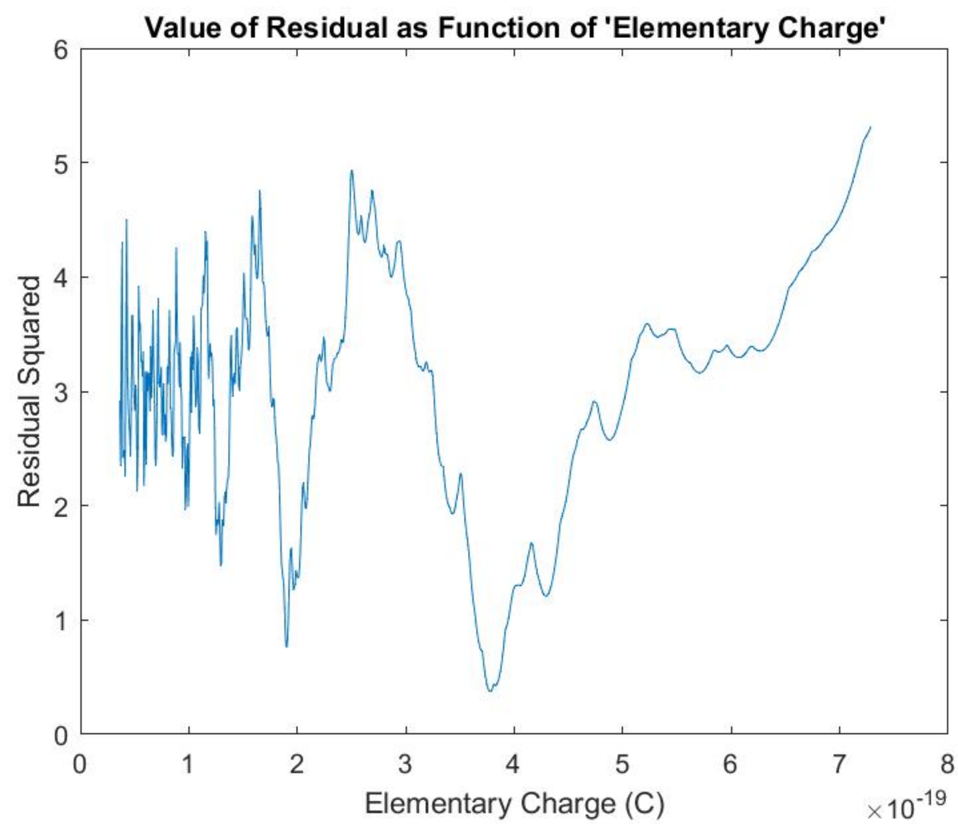

Figure I: Value of Residual Squared for Test Fundamental Charges Ranging From $3.64 * 10^{-21}$ to $6.86 * 10^{-18}$ Coulombs. Plot window reduced from full range tested for ease of viewing (see **Figure III** in the Appendix for full range).

Of course, each value of q has an associated error δq. As such, the equation we are actually working with is $q + \delta q = n(e + \delta e)$. While δq is known via propagation of error, assigning a value to δe is more difficult.

Given there were 36 drops measured in this experiment, and noting the similarity between this problem and fitting a line through the origin, a good estimate for the error is $\delta e = e * \sqrt{\frac{S^2}{N-1}} = 3.9 * 10^{-20}$, where N= 36 (Taylor, 1999). Therefore, the calculated value for the fundamental electric constant is $e = (3.8 \pm 0.4 * 10^{-19})\ C$.
Using this value of e, the fit to the different values of q was plotted and seen in **Figure II** below:

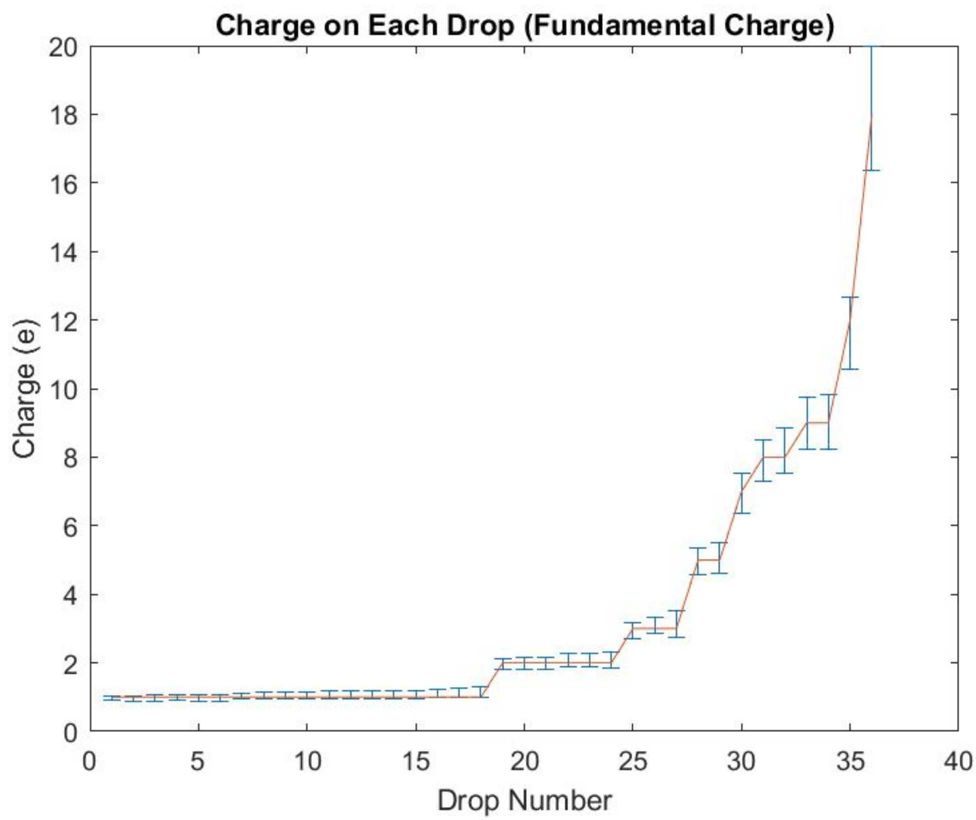

Figure II: Measured values of q in terms of the measured fundamental constant e for each drop sorted from smallest to largest, with error bars shown. Values of $\frac{q}{e}$ were rounded to the nearest integer, and all were within the bounds of error.

Therefore, minimizing the squared residual values produces a fit that allows all values of $\frac{q}{e}$ rounded to the nearest integer to fit within the bounds of error.

Conclusions

The defined value of the fundamental electric charge, $e \equiv 1.602176634 * 10^{-19}$, is far outside the error bound of $e_{measured} = (3.8 \pm 0.4) * 10^{-19}$, so there is clearly either significant systematic error or an issue with the data analysis. Interestingly, the value with the second lowest residual of 0.757 is $e' = 1.90 * 10^{-19}$, which is much closer to the tabulated value, and given the error discussed in the **Results** section, $1.90 * 10^{-19} * \sqrt{\frac{0.757}{35}} = 2.8 * 10^{-20}$, which would put this value on the bound of random error. However, this value has more than double the squared residual, and as such does not come nearly as close to having all values of q be an integer value of e, which further suggests some systematic error affecting the experiment.

As this experiment has been performed many times by different individuals since the 1900's, the many different possible sources of error have been discussed in the literature. The oil particles not perfectly obeying Stokes Law has already been at least partially corrected for. However, given the extreme sensitivity of small particles to effects such as Brownian motion, and given the number of particles with a very small radius, the current viscosity correction may not be sufficient. Indeed, it has been suggested that due to this issue (Kapusta, 1975) the best fall time for the oil drop is approximately 10 seconds, whereas most drops in this experiment fell far longer.

Other potential sources of error that are generally considered not as large, but were much more difficult to detect and correct for (Millikan, 1913). These include the presence of convection, as well as the non-uniformity of the electric field near the boundaries and around other charged particles. Furthermore, while the treatment of finding a common divisor of all values of q is currently a simple minimization problem, a more rigorous analysis such as a method to find the approximate common divisor (Galbraith, Gebregiyorgis, & Murphy, 2016) might be able to eke out a more accurate answer from the same data.

Sources Cited

Allen, M. D., & Raabe, O. G. (1985). Slip correction measurements of spherical solid aerosol particles in an improved Millikan apparatus. Aerosol Science and Technology, 4(3), 269–286. https://doi.org/10.1080/02786828508959055

Foot, R., Lew, H., & Volkas, R. R. (1993). Electric-charge quantization. Journal of Physics G: Nuclear and Particle Physics, 19(3), 361–372. https://doi.org/10.1088/0954-3899/19/3/005

Galbraith, S. D., Gebregiyorgis, S. W., & Murphy, S. (2016). Algorithms for the approximate common divisor problem. LMS Journal of Computation and Mathematics, 19(A), 58–72. https://doi.org/10.1112/s1461157016000218

Griffiths, D. J. (1999). Electrostatics. In Introduction to Electrodynamics (3rd ed., pp. 103–106). essay, Cambridge University Press.

Hogan, Benjamin and Hasbun, Javier E. (2016) "The Millikan Oil Drop Experiment: A Simulation Suitable For Classroom Use," Georgia Journal of Science, Vol. 74, No. 2, Article 7. Available at: https://digitalcommons.gaacademy.org/gjs/vol74/iss2/7

Kapusta, J. I. (1975). Best measuring time for a Millikan oil drop experiment. American Journal of Physics, 43(9), 799–800. https://doi.org/10.1119/1.9710

Laughlin, R. B. (1983). Anomalous Quantum Hall Effect: An incompressible quantum fluid with fractionally charged excitations. Physical Review Letters, 50(18), 1395–1398. https://doi.org/10.1103/physrevlett.50.1395

Millikan, R. A. (1911). The isolation of an ion, a precision measurement of its charge, and the correction of stokes's law. Physical Review (Series I), 32(4), 349–397. https://doi.org/10.1103/physrevseriesi.32.349

Millikan., R. A. (1913). On the elementary electrical charge and the Avogadro constant. Physical Review, 2(2), 109–143.

Pavlis, N. K., Holmes, S. A., Kenyon, S. C., & Factor, J. K. (2012). The development and evaluation of the Earth Gravitational Model 2008 (EGM2008). Journal of Geophysical Research: Solid Earth, 117(B4). https://doi.org/10.1029/2011jb008916 https://doi.org/10.1103/physrev.2.109

Taylor, J. R. (1997). An introduction to error analysis: the study of uncertainties in physical measurements (2nd ed.). Mill Valley Ca.: University Science Books.

Tiesinga, E. , Mohr, P. , Newell, D. and Taylor, B. (2021), CODATA Recommended Values of the Fundamental Physical Constants: 2018, Reviews of Modern Physics, [online], https://doi.org/10.1103/RevModPhys.93.025010

Appendix

The screen segment spacing was measured to be $(4.19 \pm 0.01) * 10^{-2}\ m$, and the wire diameter on the screen was measured as $(2.87 \pm 0.03) * 10^{-2}\ m$. Given the wire diameter is $(2.66 \pm 0.02) * 10^{-2}\ m$, the and as there distance the oil dropped was 6 segments, the fall distance was $(2.30 \pm 0.05) * 10^{-3}\ m$. Given this, the relevant values of each drop are given in **Table II** below:

#	Volts (V)	Time (s)	Velocity ($mm\ s^{-1}$)	Radius (μm)	Mass ($10^{-14} kg$)	Charge ($10^{-18}\ C$)
1	44.6 ± 0.5	18.5 ± 0.4	1.26 ± 0.05	0.99 ± 0.03	0.40 ± 0.003	4.4 ± 0.04
2	107.0 ± 0.5	13.5 ± 0.3	0.172 ± 0.007	1.17 ± 0.03	0.65 ± 0.005	3.0 ± 0.02
3	207 ± 5	33.8 ± 0.7	0.069 ± 0.003	0.72 ± 0.02	0.15 ± 0.001	3.4 ± 0.04
4	86 ± 2	14.3 ± 0.2	0.163 ± 0.006	1.13 ± 0.02	0.60 ± 0.004	3.4 ± 0.03
5	111 ± 2	32 ± 1	0.073 ± 0.004	0.74 ± 0.03	0.17 ± 0.002	0.75 ± 0.009
6	94 ± 2	19.8 ± 0.7	0.118 ± 0.007	0.96 ± 0.03	0.36 ± 0.004	1.9 ± 0.02
7	101.8 ± 0.6	15.17 ± 0.07	0.153 ± 0.004	1.10 ± 0.02	0.55 ± 0.003	2.6 ± 0.02
8	78 ± 2	57 ± 2	0.041 ± 0.002	0.55 ± 0.02	0.067 ± 0.0007	0.42 ± 0.006
9	133 ± 5	44.6 ± 0.4	0.052 ± 0.002	0.62 ± 0.01	0.099 ± 0.0006	0.36 ± 0.003
10	110.8 ± 0.6	32 ± 1	0.073 ± 0.005	0.74 ± 0.03	0.17 ± 0.002	0.75 ± 0.009
11	179 ± 2	37.0 ± 0.9	0.063 ± 0.003	0.69 ± 0.02	0.13 ± 0.001	0.37 ± 0.003
12	145 ± 4	39.5 ± 0.6	0.059 ± 0.002	0.66 ± 0.01	0.121 ± 0.0008	0.41 ± 0.004
13	178 ± 2	33.2 ± 1	0.070 ± 0.004	0.73 ± 0.03	0.16 ± 0.002	0.43 ± 0.005
14	228 ± 1	19.6 ± 0.4	0.119 ± 0.005	0.96 ± 0.02	0.36 ± 0.003	0.78 ± 0.007
15	110 ± 3	48 ± 1	0.049 ± 0.002	0.60 ± 0.02	0.089 ± 0.0007	0.40 ± 0.004
16	129 ± 1	41 ± 1	0.056 ± 0.003	0.65 ± 0.02	0.11 ± 0.001	0.41 ± 0.005
17	228 ± 1	30.2 ± 0.3	0.077 ± 0.002	0.77 ± 0.01	0.19 ± 0.001	0.40 ± 0.003
18	168 ± 1	36 ± 2	0.065 ± 0.004	0.70 ± 0.03	0.14 ± 0.002	0.41 ± 0.005
19	111 ± 1	31.0 ± 0.5	0.075 ± 0.003	0.76 ± 0.02	0.18 ± 0.001	0.78 ± 0.007
20	117 ± 2	17.1 ± 0.3	0.137 ± 0.005	1.03 ± 0.02	0.45 ± 0.003	0.19 ± 0.02
21	199 ± 2	21.4 ± 0.1	0.109 ± 0.003	0.92 ± 0.02	0.32 ± 0.002	0.78 ± 0.005
22	304.6 ± 0.6	24.9 ± 0.5	0.094 ± 0.004	0.85 ± 0.02	0.25 ± 0.002	0.40 ± 0.003
23	234.1 ± 0.6	7.5 ± 0.2	0.310 ± 0.02	1.58 ± 0.04	1.6 ± 0.01	3.4 ± 0.03
24	157 ± 2	39.9 ± 0.6	0.058 ± 0.002	0.66 ± 0.02	0.12 ± 0.008	0.37 ± 0.003
25	130 ± 1	22.4 ± 0.2	0.104 ± 0.003	0.90 ± 0.02	0.30 ± 0.002	1.11 ± 0.008
26	205 ± 1	5.2 ± 0.2	0.45 ± 0.02	1.91 ± 0.06	2.9 ± 0.03	6.86 ± 0.07
27	173.9 ± 1	18.1 ± 0.1	0.129 ± 0.004	1.00 ± 0.02	0.42 ± 0.002	6.86 ± 0.007
28	232 ± 1	8.0 ± 0.1	0.29 ± 0.01	1.53 ± 0.03	1.5 ± 0.01	3.1 ± 0.02
29	163 ± 1	37.8 ± 0.7	0.062 ± 0.003	0.68 ± 0.02	0.13 ± 0.001	0.39 ± 0.003
30	196 ± 2	33 ± 1	0.071 ± 0.004	0.73 ± 0.03	0.16 ± 0.002	0.40 ± 0.005
31	165 ± 2	38.8 ± 0.6	0.06 ± 0.002	0.67 ± 0.02	0.124 ± 0.0009	0.37 ± 0.003
32	305 ± 1	24.8 ± 0.4	0.094 ± 0.004	0.85 ± 0.02	0.25 ± 0.002	0.41 ± 0.003
33	221 ± 1	31.0 ± 0.7	0.075 ± 0.003	0.76 ± 0.02	0.18 ± 0.001	0.39 ± 0.003
34	108 ± 1	32 ± 1	0.072 ± 0.004	0.74 ± 0.03	0.17 ± 0.001	0.75 ± 0.009
35	114 ± 1	49 ± 2	0.048 ± 0.003	0.59 ± 0.02	0.086 ± 0.0009	0.37 ± 0.004
36	128 ± 1	22 ± 1	0.107 ± 0.007	0.91 ± 0.03	0.31 ± 0.004	1.2 ± 0.01

Table II: Average value and standard deviation of balance voltage, fall time, terminal velocity, radius, mass, and charge for each drop.

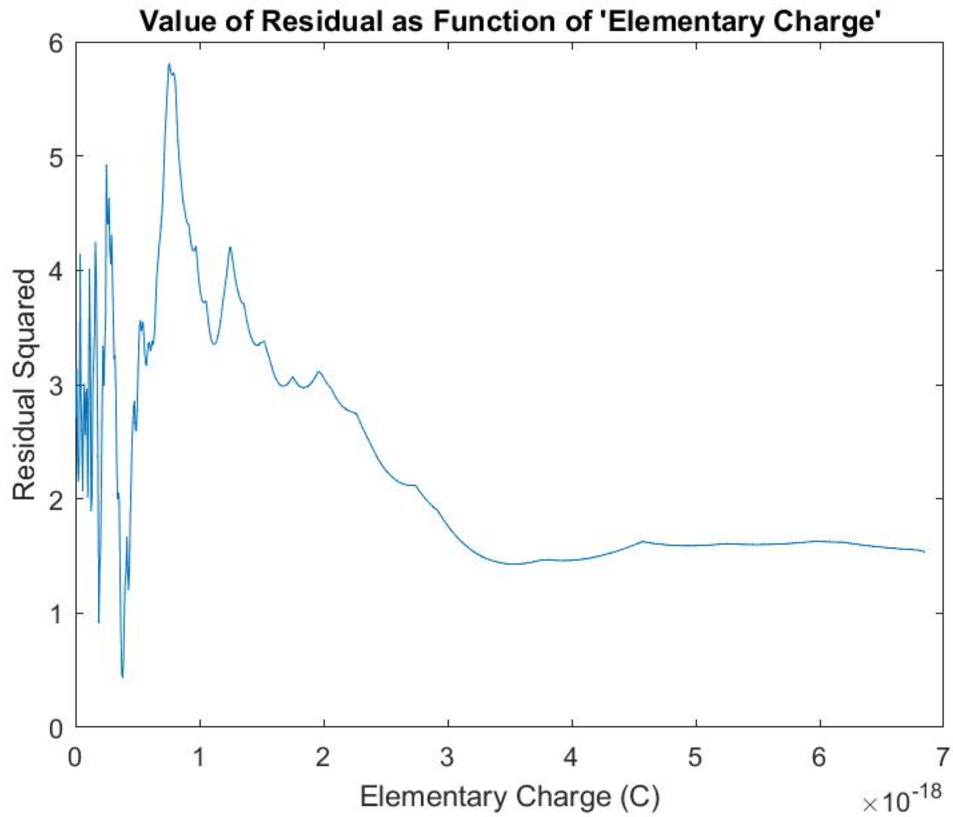

Figure III: Full range of **Figure II**